線形代数及び演習II 宿題プリント No.10 解答 (12.16)

1.

$$\boldsymbol{v}_1=\begin{pmatrix}1\\2\\1\end{pmatrix},\quad \boldsymbol{v}_2=\begin{pmatrix}1\\0\\1\end{pmatrix},\quad \boldsymbol{v}_3=\begin{pmatrix}1\\1\\2\end{pmatrix}$$ とおく.

$$\boldsymbol{u}_1=\frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}=\begin{pmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{pmatrix},\quad \tilde{\boldsymbol{v}}_2=\boldsymbol{v}_2-(\boldsymbol{v}_2,\boldsymbol{u}_1)\boldsymbol{u}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}-\frac{2}{\sqrt{6}}\begin{pmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}=\begin{pmatrix}2/3\\-2/3\\2/3\end{pmatrix}$$

$$\boldsymbol{u}_2=\frac{\tilde{\boldsymbol{v}}_2}{\|\tilde{\boldsymbol{v}}_2\|}=\frac{\sqrt{3}}{2}\begin{pmatrix}2/3\\-2/3\\2/3\end{pmatrix}=\begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\quad \tilde{\boldsymbol{v}}_3=\boldsymbol{v}_3-(\boldsymbol{v}_3,\boldsymbol{u}_1)\boldsymbol{u}_1-(\boldsymbol{v}_3,\boldsymbol{u}_2)\boldsymbol{u}_2$$

$$=\begin{pmatrix}1\\1\\2\end{pmatrix}-(5/\sqrt{6})\begin{pmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}-2/\sqrt{3}\begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}=\begin{pmatrix}1-\frac{5}{6}-\frac{2}{3}\\1-\frac{10}{6}+\frac{2}{3}\\2-\frac{5}{6}-\frac{2}{3}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}$$

$$\boldsymbol{u}_3=\frac{\tilde{\boldsymbol{v}}_3}{\|\tilde{\boldsymbol{v}}_3\|}=\sqrt{2}\begin{pmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-\frac{\sqrt{2}}{2}\\0\\\frac{\sqrt{2}}{2}\end{pmatrix}\left(=\begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}\right)$$

よって 正規直交化された基は

$$\left\{\begin{pmatrix}\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix},\begin{pmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix},\begin{pmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}\right\}$$

2.

$$\det(A-\lambda I)=\begin{vmatrix}3-\lambda & 2\\2 & 3-\lambda\end{vmatrix}=(3-\lambda)^2-4=(\lambda-5)(\lambda-1)=0$$

$\lambda=5,1$ ∴ 固有値

$\lambda=5$ のとき $(A-5I)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2&2\\2&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ より $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}c$ $(c\neq0)$

$\lambda=1$ のとき $(A-I)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2&2\\2&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ より $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}c$ $(c\neq0)$

よって 直交行列 $P=\begin{pmatrix}1/\sqrt{2}&-1/\sqrt{2}\\1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}$ により $P^TAP=\begin{pmatrix}5&0\\0&1\end{pmatrix}$ と対角化される.

$\Big($ $P=\begin{pmatrix}-1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}$ あるいは $P=\begin{pmatrix}1/\sqrt{2}&1/\sqrt{2}\\-1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}$ ととれば

$P^TAP=\begin{pmatrix}1&0\\0&5\end{pmatrix}$ $\Big)$