Lyapunov function and spectrum comparison for a reaction-diffusion system with mass conservation

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Abstract

We are dealing with a two-component system of reaction-diffusion equations with mass conservation in a bounded domain with the Neumann boundary conditions. We prove the global boundedness of the solution in $L^\infty$-norm for $t \geq 0$ under a condition, and then the existence of a Lyapunov function. Moreover, by studying the linearized eigenvalue problem of a nonconstant equilibrium solution, we provide a comparison theorem for the spectrum between the linearized operators of the system and an appropriate nonlocal scalar equation. As an application of the comparison result we obtain that any stable equilibrium solution must be monotone if the space dimension is one. It is also shown that a modified system with a new parameter, which covers the present model, possess a Lyapunov function.

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1 Introduction

We consider the following system of reaction-diffusion equations:

\[
\begin{aligned}
    u_t &= d\Delta u - g(u + v) + kv, \\
    v_t &= \Delta v + g(u + v) - kv,
\end{aligned}
\quad (x \in \Omega)
\]  

(1.1)

with the Neumann boundary conditions

\[
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad (x \in \partial \Omega),
\]  

(1.2)

where \(d, k\) are positive constants, \(g(u, v)\) is sufficiently smooth function and \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with the smooth boundary \(\partial \Omega\).

We may assume

\[
g(s)/s \leq K \quad (s > 0), \quad \lim_{s \to +0} \frac{g(s)}{s} = \beta > 0, \quad g \in C^1([0, \infty); \mathbb{R}),
\]  

(1.3)

which ensure the nonnegativity of the solution of the initial value problem provided \(u(x, 0), v(x, 0) \geq 0\) in \(\Omega\) by the maximum principle applied to each equation of the system.

Since the total mass

\[
\int_{\Omega} \{u(x, t) + v(x, t)\} dx
\]

is conserved in time evolution for the solution \((u(x, t), v(x, t))\) of (1.1), we fix the condition as

\[
s := \langle u(\cdot, 0) \rangle + \langle v(\cdot, 0) \rangle,
\]  

(1.4)

where we use the notation

\[
\langle f(\cdot) \rangle := \frac{1}{|\Omega|} \int_{\Omega} f(x) dx, \quad |\Omega| : \text{the volume of } \Omega.
\]

A specific example of \(g\) is given by

\[
g(w) = \frac{kw}{(aw + 1)^2}
\]

in [10] and [14] as a simple model exhibiting similar dynamical behaviors to those observed in a more complicated model describing the cell polarity. In fact, for such a function \(g\) they show the Turing type instability for appropriate parameter values \(d, k\) and \(s\). They also exhibit by a numerical simulation (under the periodic boundary conditions) that after an emergence of a wave pattern and a long transient behavior, the solution converges to a stationary solution with the shape of a single spike. By virtue of this concentration property their model equations could be helpful for the understanding of the cell polarity phenomenon. However,
they don’t show why such a spike pattern is stable while the multi-spike pattern is unstable.

The purpose of this article is to reveal some dynamical property and the stability of equilibrium solution of (1.1)-(1.2), which justifies the numerical observation about the asymptotic profile of the solution. More precisely, we show that the system of (1.1) allows a Lyapunov function, by which we can assert that the $\omega$-limit set of any bounded orbit of the solution consists of equilibrium solutions. In addition, by investigating the linearized eigenvalue problem for any equilibrium solution, we see that the profile of a stable equilibrium solution in one-dimensional space must be monotone (or unimodal, i.e., of a single spike if the periodic boundary conditions are assumed).

We remark that for $f(u)$ instead of $g(u + v)$ the similar dynamical property is also found in the same literatures [10], [14]. For some mathematical result justifying it is proposed in [13], [12] by converting the equations to the equations of the phase field type system as in [2] and [5]. The present article is thereby a successive work for a class of conserved reaction-diffusion systems.

We also remark on some results for the general system

$$u_t = du_{xx} - f(u, v), \quad v_t = v_{xx} + f(u, v),$$

which arises as a simple chemical conserved reaction model, called a closed reaction-diffusion system. The existence of traveling waves for a specific nonlinear term is shown in [3] and [4]. Moreover, this type of the system is used in modeling of a precipitation kinetics, and the formation of Liesegang patterns is shown in [9] and [17]. As seen in [15], by putting $w = u + v, v = v$, the above system can be converted to the one belonging to a class of the systems

$$w_t = [a(u, v)w_x + b(u, v)v_x]_x, \quad v_t = v_{xx} + \tilde{f}(u, w).$$

Under some conditions on $a(u, v)$ and $b(u, v)$ the instability of a spike solution of the system in the infinite interval or a sufficiently large interval with the Neumann boundary conditions is investigated in [15] (see also [16] on the stability of layer solutions).

Coming back to the present system, we state our results precisely. We first provide the global boundedness of the solution for $t \geq 0$ under the condition (1.3).

**Proposition 1.1** Assume (1.3) and that $u_0, v_0 \in C^0(\overline{\Omega}, \mathbb{R}), u_0(x), v_0(x) \geq 0$. Let $(u(x, t), v(x, t))$ be a solution of (1.1) with (1.2) and $(u(x, 0), v(x, 0)) = (u_0(x), v_0(x))$. Then $\|u(\cdot, t)\|_{L^\infty}, \|v(\cdot, t)\|_{L^\infty}, \|\partial u(\cdot, t)/\partial x_j\|_{L^\infty}, \|\partial v(\cdot, t)/\partial x_j\|_{L^\infty} (j = 1, 2, \ldots)$ are uniformly bounded in $t \geq 0$. Moreover, there is a positive $M > 0$, independent of the initial data, such that

$$\limsup_{t \to \infty} \left[ \|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} + \sum_{j=1}^{n} \left\{ \left\| \frac{\partial u(\cdot, t)}{\partial x_j} \right\|_{L^\infty} + \left\| \frac{\partial v(\cdot, t)}{\partial x_j} \right\|_{L^\infty} \right\} \right] \leq Ms$$

holds.
Next we show the system allows a Lyapunov function. Define the functional

\[ E(u,v) := \int_\Omega \left\{ \frac{d}{2} |\nabla(u+v)|^2 + \frac{1}{2} |\nabla(du+v)|^2 + Q(u+v) \right\} \, dx, \]

(1.5)

\[ Q(w) := (1-d) \int w g(z) \, dz + \frac{dk}{2} w^2. \]

Then we have

**Proposition 1.2** For a smooth solution \((u(x,t), v(x,t))\) to (1.1) with (1.2) (or the periodic boundary conditions),

\[ \frac{d}{dt} E(u(\cdot,t), v(\cdot,t)) = -\int_\Omega [(1+d)(u_t+v_t)^2 + k|\nabla(du+v)|^2] \, dx \leq 0 \]

(1.6)

holds. Then \(dE(u(\cdot,t), v(\cdot,t))/dt = 0 \quad (\forall t \in \mathbb{R})\) implies that \((u(x,t), v(x,t))\) is an equilibrium solution.

The proof of Propositions 1.1 and 1.2 will be given in §2.

We remark that if (1.3) is satisfied, then by the propositions, the orbit of the solution in an appropriate phase space is precompact and the \(\omega\)-limit set of the orbit consists of equilibrium solutions (see for instance [7] or [8]). In other words, the asymptotic state of any solution to (1.1) with (1.2) and (1.3) is determined by the stationary problem.

Before going to the stationary problem, we covert the system by putting \(w = u + v\) as

\[
\begin{aligned}
w_t &= d\Delta w + (1-d)\Delta v, \\
v_t &= \Delta v + g(w) - kv, \\
\end{aligned} \quad (x \in \Omega)
\]

(1.7)

with the Neumann boundary conditions and \(\langle w(\cdot,t) \rangle = s\) for \(t \geq 0\).

We decompose the variables under the condition \(d \neq 1\) as

\[ \begin{cases} w = \langle w \rangle + W = s + W, \\ v = \langle v \rangle + V = \xi + V \end{cases} \]

(1.8)

Then (1.7) is decomposed as

\[ W_t = d\Delta W + (1-d)\Delta V, \]

(1.9)

\[ V_t = \Delta V + g(s+W) - \langle g(s+W) \rangle - kV, \]

(1.10)

\[ \xi_t := \langle g(s+W) \rangle - k\xi \]

(1.11)

with

\[ \langle V(\cdot,t) \rangle = \langle W(\cdot,t) \rangle = 0. \]

(1.12)
namely the solutions \((w(x,t), v(x,t))\) to \((1.7)\) and \((W(x,t), V(x,t), \xi(t))\) to \((1.10)-(1.11)\) have the 1-1 correspondence by

\[
(w(x,t), v(x,t)) = (s + W(x,t), \xi(t) + V(x,t)), \quad \xi(t) = \langle v(\cdot, t) \rangle,
\]

consequently, the solution \((u(x,t), v(x,t))\) of \((1.1)\) with \((1.2)\) and \((1.4)\) is given by

\[
(u(x,t), v(x,t)) = (s - \xi(t) + W(x,t) - V(x,t), \xi(t) + V(x,t)).
\]

We notice that the equations for \(W\) and \(V\) of \((1.9)\) and \((1.10)\) are independent of \(\xi\), therefore it suffices to consider \((1.9)\) and \((1.10)\) with the Neumann conditions and \((1.12)\) unless \(d = 1\). If \(d = 1\), then the equations turn to be quite simple, we thereby exclude this case in the present study.

Now consider the stationary problem, that is,

\[
\begin{aligned}
d\Delta W + (1 - d)\Delta V &= 0, \\
\Delta V + g(s + W) - \langle g(s + W) \rangle - kV &= 0, \quad (x \in \Omega),
\end{aligned}
\]

with

\[
\frac{\partial W}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 \quad (x \in \Omega), \quad \langle W \rangle = \langle V \rangle = 0.
\]

By the second equation of \((1.13)\)

\[
dW + (1 - d)V = 0
\]

follows. Hence, we see that \((1.13)\) with \((1.14)\) is equivalent to the next scalar equation

\[
\begin{aligned}
\Delta W - \alpha(g(s + W) - \langle g(s + W) \rangle) - kW &= 0 \quad (x \in \Omega), \\
\frac{\partial W}{\partial \nu} &= 0 \quad (x \in \partial \Omega), \quad \langle W \rangle = 0,
\end{aligned}
\]

where we put

\[
\alpha := \frac{1 - d}{d} \quad (d \neq 1).
\]

In fact if \(W^*\) is a solution of \((1.15)\) with \((1.16)\), then \((W^*, V^*) = (W^*, -W^*/\alpha)\) gives a solution to \((1.13)\) with the Neumann boundary conditions, thus an equilibrium solution of \((1.1)\) with \((1.2)\) and \((1.4)\) is obtained by

\[
(u^*(x), v^*(x)) = \left( s - \frac{1}{k} \langle g(s + W^*) \rangle + \frac{1}{1 - d} W^*(x), \frac{1}{k} \langle g(s + W^*) \rangle - \frac{d}{1 - d} W^*(x) \right)
\]

The next result is on the spectrum comparison between the linearized operator at an equilibrium solution in the system and that of the above scalar equation.
(1.15)-(1.16). The idea of the comparison is due to [1] for the Cahn-Hilliard equation and the phase field system. Later, some improvement for the phase field system is proposed in [12]. We will develop the study to our system. To state the result, we introduce function spaces. Denote by $L^2(\Omega)$ the space of square integrable functions with the norm $\|u\|$ ($u \in L^2(\Omega)$) and

$$H^m(\Omega) = \{u \in L^2(\Omega) : \|u\|_H^m := (\|u\|^2 + \sum_{|a| \leq m} \|\partial^a u/\partial x^a\|^2)^{1/2} < \infty\},$$

where

$$a = (a_1, \ldots, a_n), \quad a_j \geq 0, \quad a_j \in \mathbb{Z}, \quad |a| = \sum_{j=1}^n a_j.$$

We also introduce a subspace of $L^2(\Omega)$ and $H^m(\Omega)$ with the average zero as

$$\overline{L}^2(\Omega) := \{u \in L^2(\Omega) : \int_\Omega u \, dx = 0\}, \quad \overline{H}^m(\Omega) := \{u \in H^m(\Omega) : u \in \overline{L}^2(\Omega)\}.$$

We let $W^*$ be a solution of (1.15) with (1.16) and put $w^* = s + W^*$. We fix $s$ and consider the linearized operator $A$ for (1.7)

$$A \left( \begin{array}{c} \phi \\ \psi \end{array} \right) := - \left( \begin{array}{c} d \Delta \phi + (1 - d) \Delta \psi \\ \Delta \psi + g'(w^*) \phi - k \psi \end{array} \right), \quad (1.17)$$

and the linearized operator $L$ at $W^*$ for (1.15)

$$L(\phi) := - \Delta \phi + \alpha (g'(w^*) \phi - \langle g'(w^*) \phi \rangle) + k \phi, \quad (1.18)$$

with the domain

$$D(A) = \{(\phi, \psi) \in \overline{H}^2(\Omega) \times H^2(\Omega) : \partial \phi/\partial \nu = \partial \psi/\partial \nu = 0 \ (x \in \partial \Omega)\},$$

and

$$D(L) = \{\phi \in \overline{H}^2(\Omega) : \partial \phi/\partial \nu = 0 \ (x \in \partial \Omega)\},$$

respectively.

We compare the spectrum of $A$ and $L$. It is easy to see that any eigenvalue is real for $L$. We let $\{\mu_j\}_{j=1,\ldots,\infty}$ be the set of all the eigenvalues of $L$ arranged in increasing order with counting the multiplicity as

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m \leq \cdots \quad (1.19)$$

On the other hand there is a possibility $A$ allows a complex eigenvalue. Fortunately, any eigenvalue $\lambda$ of $A$ is real if the real part of $\lambda$ is less than $k/2$ (see §4), in the sequel it is possible to compare all the real eigenvalues which are related to the stability of the solutions. We thereby let $\{\lambda_j\}_{j=1,\ldots,N}$ be the set of all the eigenvalues of $L$ less than $k/2$, arranged in increasing order with counting the multiplicity as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N < k/2 \quad (1.20)$$

The next theorem gives the spectrum comparison between the linearized operators.
Theorem 1.3 The number of the negative eigenvalues of $A$ and $L$ coincides, and so the multiplicity of the zero eigenvalue of both the operators. Moreover,

$$|\lambda_j| \leq \frac{d}{1+d} |\mu_j| \quad (j = 1, 2, \ldots, N)$$  \hspace{1cm} (1.21)

holds, where $\mu_j$ and $\lambda_j$ are as in (1.19) and (1.20) respectively.

We give a remark that (1.15) with (1.16) is the Euler-Lagrange equation of

$$G(w) := \int_{\Omega} \left\{ \frac{d}{2} |\nabla w|^2 + Q(w) \right\} dx,$$  \hspace{1cm} (1.22)

in the space $\overline{H}^1(\Omega)$, where $Q$ is as in (1.5). By virtue of Theorem 1.3, noticing $(L(\phi), \phi)'_{L^2}$ is the second variation of $G(w)$, we can apply the stability result for (1.22) in [6] to the system (1.1) (see also [19], and as for the periodic boundary conditions see [12]).

Corollary 1.4 Consider the one-dimensional case, namely, the case that $\Omega$ is interval. Let $(u(x), v(x))$ be a stable equilibrium solution to (1.1) with (1.2) and (1.4). Then $u(x)$ and $v(x)$ are monotone, namely, those are constant or strictly monotone. If the boundary conditions are periodic, then $u(x)$ and $v(x)$ are constant or unimodal.

We state the idea of the proof of Theorem 1.3 in brief. Noticing that the equations of the eigenvalue problem of $A$ can be written in a single equation for $\phi$, we introduce a nonlocally weighted eigenvalue problem for $L$ with a free parameter $\beta \geq 0$ and establish the comparison between the eigenvalues of the nonlocal problem and the original $L$, which is the case $\beta = 0$ up to multiplication of the constant $d/(1 + d)$. By the continuity property of the spectra in $\beta$ each eigenvalue of $A$ less than $k/2$ is given by choosing an appropriate $\beta > 0$. By this we obtain the desired assertion.

In the next section we prove Propositions 1.1 and 1.2. Then in §3 we discuss the nonlocally weighted eigenvalue problem and prove Theorem 1.3 by the key lemmas, which will be proved in §4.

In §5 we consider a modified system which is given by $g(u + \gamma v)$ with the parameter $\gamma \in [0, 1]$, instead of $g(u + v)$. This model system covers the present case. We show that the new system also possesses a Lyapunov function for any $\gamma \in [0, 1]$. We conclude the paper by an appendix in which it is shown that the Turing-type instability certainly takes place for a specific $g = kw/(aw + 1)^2$.

2 Proofs of Propositions 1.1 and 1.2

2.1 Proof of Proposition 1.1

We simply write the equations of (1.1) as

$$u_t = d\Delta u - f(u, v), \quad v_t = \Delta v + f(u, v).$$
It is known that the solutions \( u(x,t) \) and \( v(x,t) \) are nonnegative and smooth in \( \Omega \times (0, \infty) \). We first prove that there are \( c_0, M_0 > 0 \) such that

\[
0 \leq u(x,t), \quad v(x,t) \leq c_0 e^{M_0 t}.
\]  

(2.1)

Indeed, from the assumption there are \( a_{ij} > 0 \) \( (i, j = 1, 2) \) for which

\[
-f(u, v) \leq a_{11} u + a_{12} v, \quad f(u, v) \leq a_{21} u + a_{22} v
\]

hold, thus, the solutions satisfy the differential inequalities

\[
u_t \leq d \Delta u + a_{11} u + a_{12} v, \quad v_t \leq \Delta v + a_{21} u + a_{22} v.
\]

With the aid of the solutions \( p(t), q(t) \) of the ODEs

\[
p_t = a_{11} p + a_{12} q, \quad q_t = a_{21} p + a_{22} q,
\]

with the initial conditions

\[
p(0) = \sup_{x \in \Omega} u_0(x) + 1, \quad q(0) = \sup_{x \in \Omega} v_0(x) + 1,
\]

we can apply the comparison argument to show

\[
0 \leq u(x,t) \leq p(t), \quad 0 \leq v(x,t) \leq q(t).
\]

Since the solutions \( p(t) \) and \( q(t) \) have at most the exponential growth, we obtain (2.1).

Before going to the proof of the global boundedness of the solutions, we prepare two lemmas.

**Lemma 2.1** There exists a constant \( K = K(\Omega) > 0 \) such that

\[
\|\phi\|_{L^\infty(\Omega)} \leq \left(\frac{1}{|\Omega|}\right)\|\phi\|_{L^1(\Omega)} + K(\Omega)\|\nabla \phi\|_{L^\infty(\Omega)}
\]

holds.

**Proof.** From the regularity of \( \partial \Omega \), there exist a constant \( c > 0 \) and a curve \( \gamma : [0, 1] \rightarrow \Omega \) with the conditions

\[
\gamma(0) = x, \quad \gamma(1) = y, \quad |\dot{\gamma}(\tau)| \leq c \quad (0 \leq \tau \leq 1)
\]

for arbitrarily given \( x, y \in \Omega \). Take any \( \phi \in C^1(\Omega) \)

\[
\phi(x) - \phi(y) = \int_0^1 \frac{d}{d\tau} \phi(\gamma(\tau)) d\tau
\]

\[
= \int_0^1 \nabla \phi(\gamma(\tau)) \cdot \dot{\gamma}(\tau) d\tau.
\]

8
Hence, we have

\[ |\phi(x)| \leq |\phi(y)| + c|\nabla \phi|_{L^1(\Omega)}. \]

Integrate both sides of this inequality in \(y\), we easily see the desired inequality follows.

Next, let \(G_1(x, y, t)\) and \(G_2(x, y, t)\) be the heat kernels for \(d\Delta\) and \(\Delta\) with the Neumann boundary condition respectively. We have the next estimate.

**Lemma 2.2** For the heat kernels \(G_j\) \((j = 1, 2)\)

\[ |\nabla_x G_j(x, y, t)| \leq \frac{c_1}{t^{(n+1)/2}} \exp(-\eta_1 t - |x - y|^2/\eta_2 t) \quad (x, y \in \Omega, t > 0, j = 1, 2) \]

hold, where \(c_1, \eta_1\) and \(\eta_2\) are positive constants.

We note \(\eta_j\) \((j = 1, 2)\) depend on \(d\).

Using \(G_1, G_2\), we have the integral expression for the equations (1.1)-(1.2), that is,

\[
\begin{align*}
  u(x, t) &= \int_{\Omega} G_1(x, y, t)u_0(y)dy \\
  &\quad - \int_0^t \int_{\Omega} G_1(x, y, t - \tau)f(u(y, \tau), v(y, \tau)) dyd\tau, \quad (2.2) \\
  v(x, t) &= \int_{\Omega} G_2(x, y, t)v_0(y) \\
  &\quad + \int_0^t \int_{\Omega} G_2(x, y, t - \tau)f(u(y, \tau), v(y, \tau)) dyd\tau. \quad (2.3)
\end{align*}
\]

Given a small constant \(\delta \in (0, 1)\), we consider the solutions for \(t \geq \delta\) and estimate \(u(t, x), v(t, x)\) in \([\delta, \infty) \times \Omega\). From (2.2) and (2.3) we see

\[
\begin{align*}
  \nabla u(t, x) &= \int_{\Omega} \nabla G_1(x, y, t)u_0(y)dy - \int_{\Omega} \int_{0}^{t-\delta} \nabla G_1(x, y, t - \tau)f(u, v)dyd\tau \\
  &\quad - \int_{t-\delta}^{t} \int_{\Omega} \nabla G_1(x, y, t - \tau)f(u, v)dyd\tau, \\
  \nabla v(t, x) &= \int_{\Omega} \nabla G_2(x, y, t)v_0(y)dy + \int_{\Omega} \int_{0}^{t-\delta} \nabla G_2(x, y, t - \tau)f(u, v)dyd\tau \\
  &\quad + \int_{t-\delta}^{t} \int_{\Omega} \nabla G_2(x, y, t - \tau)f(u, v)dyd\tau.
\end{align*}
\]

where we simply write \(f(u, v) = f(u(y, \tau), v(y, \tau))\).

Recall \(w(t, x) = u(t, x) + v(t, x)\) and

\[ \int_{\Omega} w(t, x)dx = |\Omega|s \quad (t \geq 0). \]
Define a function
\[ \rho(t) := \sup \{ w(x, \tau) : 0 \leq \tau \leq t, x \in \Omega \} \quad (2.4) \]

**Lemma 2.3** There is a small number \( \delta > 0 \), which is independent of the initial data, such that the function \( \rho \) defined by (2.4) satisfies
\[ 0 \leq \rho(t) \leq \max \{ Ms, \rho(\delta) \} \quad (t \geq 0), \]
where \( M \) depends on \( c_1, \eta_1, \eta_2, \delta, \Omega \) and \( a_{ij} \) (1 \( \leq i, j \leq 2 \)) but independent of \( s \).

**Proof.** We first estimate the solutions in the region \( t \geq \delta \).

\[
| \nabla u(x, t) | \\
\leq \int_{\Omega} | \nabla G_1(x, y, t) | | u_0(y) | dy + \int_{0}^{t-\delta} \int_{\Omega} | \nabla G_1(t - \tau, x, y) | | f(u, v) | dy d\tau \\
+ \int_{t-\delta}^{t} \int_{\Omega} | \nabla G_1(t - \tau, x, y) | | f(u, v) | dy d\tau \\
\leq \frac{c_1}{\delta^{(n+1)/2}} \int_{\Omega} | u_0(y) | dy + \int_{0}^{t-\delta} \int_{\Omega} \frac{c_1 e^{-\eta_1 (t-\tau)}}{\delta^{(n+1)/2}} (a_{11} u(y, \tau) + a_{12} v(y, \tau)) dy d\tau \\
+ \int_{t-\delta}^{t} \int_{\Omega} \frac{c_1}{(t-\tau)^{(n+1)/2}} \exp \left( -\frac{|x-y|^2}{\eta_2 (t-\tau)} \right) (a_{11} u(y, \tau) + a_{12} v(y, \tau)) dy d\tau \\
\leq \frac{c_1 \| u_0 \|_{L^1}}{\delta^{(n+1)/2}} + \frac{c_1 s |\Omega|}{\delta^{(n+1)/2} \eta_1} \max(a_{11}, a_{12}) + \int_{t-\delta}^{t} (\pi \eta_2)^{n/2} \frac{c_1 d\tau}{\sqrt{t-\tau}} \max(a_{11}, a_{12}) \rho(t) \\
\leq \frac{c_1 \| u_0 \|_{L^1}}{\delta^{(n+1)/2}} + \frac{c_1 s |\Omega|}{\delta^{(n+1)/2} \eta_1} \max(a_{21}, a_{22}) + 2c_1 (\pi \eta_2)^{n/2} \sqrt{\delta} \max(a_{11}, a_{12}) \rho(t).
\]

Similarly, we obtain
\[
| \nabla v(x, t) | \\
\leq \frac{c_1 \| v_0 \|_{L^1}}{\delta^{(n+1)/2}} + \frac{c_1 s |\Omega|}{\delta^{(n+1)/2} \eta_1} \max(a_{21}, a_{22}) + 2c_1 (\pi \eta_2)^{n/2} \sqrt{\delta} \max(a_{21}, a_{22}) \rho(t).
\]

By Lemma 2.1, we have
\[
0 \leq u(x, t) \leq \frac{1}{|\Omega|} \| v(\cdot, t) \|_{L^1} + K(\Omega) \left\{ \frac{c_1}{\delta^{(n+1)/2}} (\| u_0 \|_{L^1} + s |\Omega| \max(a_{11}, a_{12}) / \eta_1) \\
+ 2c_1 (\pi \eta_2)^{n/2} \sqrt{\delta} \max(a_{11}, a_{12}) \rho(t) \right\},
\]
\[
0 \leq v(x, t) \leq \frac{1}{|\Omega|} \| u(\cdot, t) \|_{L^1} + K(\Omega) \left\{ \frac{c_1}{\delta^{(n+1)/2}} (\| v_0 \|_{L^1} + s |\Omega| \max(a_{21}, a_{22}) / \eta_1) \\
+ 2c_1 (\pi \eta_2)^{n/2} \sqrt{\delta} \max(a_{21}, a_{22}) \rho(t) \right\}
\]
for \( t \geq \delta, x \in \Omega \). We choose \( \delta > 0 \), which is independent of the initial data, so that
\[
2K(\Omega)c_1 (\pi \eta_2)^{n/2} \sqrt{\delta} \max(a_{ij} : 1 \leq i, j \leq 2) \leq 1/4 \quad (2.5)
\]
is satisfied. Then

\[ 0 \leq w(x, t) \leq s + \frac{K(\Omega)c_1s}{\eta_1\delta^{(n+1)/2}}(\eta_1 + 2|\Omega| \max(a_{ij} : 1 \leq i, j \leq 2)) + \frac{1}{2} \sup_{x \in \Omega, 0 \leq t \leq \tau} w(x, \tau) \]

for \( t \geq \delta \). By considering the non-blowup of the solutions in a finite time, a simple argument leads us to

\[ 0 \leq \rho(t) \leq \max \left\{ 2s + \frac{2K(\Omega)c_1s}{\eta_1\delta^{(n+1)/2}}(\eta_1 + 2|\Omega| \max(a_{ij} : 1 \leq i, j \leq 2)), \rho(\delta) \right\} \]

for \( t \geq 0 \). It is clear by (2.5) that \( \delta \) is chosen independently of the initial data. Consequently, we obtain the desired assertion of the lemma.

By virtue of Lemma 2.3 we have the global boundedness of the orbit defined by the solutions \((u(t, x), v(t, x))\) in \( L^\infty(\Omega)^2 \). Applying this result to the estimating of \(|\nabla u(x, t)|\) and \(|\nabla v(x, t)|\) with the aid of the integral equations, we easily get the global \( C^1 \) estimates for \((u(t, x), v(t, x))\), in consequence, we conclude the proof of Proposition 1.1.

### 2.2 Proofs of Proposition 1.2

We prove Proposition 1.2. Let \((u(\cdot, t), v(\cdot, t))\) be a solution of (1.1) with (1.2). A straightforward computation shows the following:

\[
\frac{d}{dt} E(u(\cdot, t), v(\cdot, t)) = \int_\Omega \left\{ d\nabla(u + v) \cdot \nabla(u + v)_t + \nabla(du + v) \cdot \nabla(du + v)_t \right. \\
\left. + [(1 - d)g(u + v) + dk(u + v)](u + v)_t \right\} dx \\
= \int_\Omega \left\{ [-d\Delta(u + v) + (1 - d)g(u + v) + dk(u + v)](u + v)_t \\
+ \nabla(du + v) \cdot \nabla(du + dv)_t + (1 - d)\nabla(du + v) \cdot \nabla v_t \right\} dx \\
= - \int_\Omega \left\{ [d\Delta(u + v) - (1 - d)g(u + v) - dk(u + v)](u + v)_t \\
+ d\Delta(du + v)(u + v)_t + (1 - d)\Delta(du + v)v_t \right\} dx \\
\]

Then we use

\[ u_t + dv_t = d\Delta(u + v) - (1 - d)g(u + v) + k(1 - d)v, \]

and

\[ u_t + v_t = \Delta(du + v), \]
to obtain
\[
\frac{d}{dt} \mathcal{E}(u(\cdot, t), v(\cdot, t)) = -\int_{\Omega} \{[u_t + dv_t - k(1 - d)v - dk(u + v)](u + v)_t \\
+ d(u_t + v_t)^2 + (1 - d)(u_t + v_t)v_t\} dx
\]
\[
= -\int_{\Omega} \{[u_t + dv_t - k(du + v) + (1 - d)v_t](u + v)_t \\
+ d(u_t + v_t)^2\} dx
\]
\[
= -\int_{\Omega} \{(1 + d)(u_t + v_t)^2 - k(du + v)\Delta(du + v)\} dx
\]
\[
= -\int_{\Omega} \{(1 + d)(u_t + v_t)^2 + k|\nabla(du + v)|^2\} dx \leq 0.
\]
Thus
\[
\frac{d}{dt} \mathcal{E}(u(\cdot, t), v(\cdot, t)) = 0 \quad (\forall t \in \mathbb{R})
\]
implies
\[
(u + v)_t = w_t = 0 \quad \text{and} \quad du + v = dw + (1 - d)v = \text{constant in } x.
\]
This yields \(W_t = V_t = 0\) in (1.9). Then the solution \(\xi(t)\) of (1.11) defined for all \(t\) must be a steady state. This concludes the proof of Proposition 1.2.

\section{Linearized eigenvalue problem}

For the operator of (1.17) we consider the eigenvalue problem
\[
\mathcal{A} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{3.1}
\]
that is,
\[
d\Delta \phi + (1 - d)\Delta \psi = -\lambda \phi, \tag{3.2}
\]
\[
\Delta \psi + g'(w^*)\phi - k\psi = -\lambda \psi. \tag{3.3}
\]
We need to prepare several lemmas to prove Theorem 1.3. The first lemma immediately follows from [11] (Theorem 6.29, Chap.3, §6-8).

\begin{lemma}
\textit{The spectrum of }\mathcal{A}\text{ consists of eigenvalues.}
\end{lemma}

\begin{lemma}
\textit{Let }\lambda\textit{ be an eigenvalue of }\mathcal{A}. \textit{If }\text{Re}\lambda < k/2\text{, then }\lambda\textit{ is real.}
\end{lemma}
Proof. Combining the two equations of (3.2)-(3.3), we have
\[d\Delta \phi - (1 - d)g'(w^*)\phi + (1 - d)k\psi = -\lambda \phi + (1 - d)\lambda \psi. \tag{3.4}\]
By putting
\[\chi = \phi + \frac{1 - d}{d} \psi \quad (i.e. \psi = \frac{d}{1 - d}(\chi - \phi)),\]
(3.2)-(3.3) are transformed to
\[d\Delta \chi = -\lambda \phi, \tag{3.5}\]
\[d\Delta \phi - (1 - d)g'(w^*)\phi + dk(\chi - \phi) = -\lambda \phi + d\lambda (\chi - \phi) \tag{3.6}\]
We prove that if \(\text{Im} \lambda \neq 0\), then \(\text{Re} \lambda \geq k/2\). Set
\[\lambda = \lambda_1 + i\lambda_2, \quad \chi = \chi_1 + i\chi_2, \quad \phi = \phi_1 + i\phi_2,\]
and
\[J_1 := \int_{\Omega}(\chi_1\phi_1 + \chi_2\phi_2)dx, \quad J_2 := \int_{\Omega}(\chi_1\phi_2 - \chi_2\phi_1)dx,\]
where \(\lambda_j \in \mathbb{R} \ (j = 1, 2)\) and \(\chi_j, \phi_j \ (j = 1, 2)\) are real-valued. Then by simple computation we obtain
\[d(\|\nabla \chi_1\|^2 + \|\nabla \chi_2\|^2) = (\lambda_1 + i\lambda_2)(J_1 + iJ_2),\]
\[d(\|\nabla \phi_1\|^2 + \|\nabla \phi_2\|^2) + d(\lambda_1 + i\lambda_2 - k)(J_1 - iJ_2)\]
\[+ (1 - d) \int_{\Omega}g'(w^*)(\phi_1^2 + \phi_2^2)dx = \{(1 + d)(\lambda_1 + i\lambda_2) - dk\}(\|\phi_1\|^2 + \|\phi_2\|^2).\]
Taking the real part and imaginary part of the equalities yields
\[d(\|\nabla \chi_1\|^2 + \|\nabla \chi_2\|^2) = \lambda_1 J_1 - \lambda_2 J_2,\]
\[\lambda_2 J_1 + \lambda_1 J_2 = 0,\]
\[d[\lambda_2 J_1 + (k - \lambda_1)J_2] = \lambda_2(1 + d)(\|\phi_1\|^2 + \|\phi_2\|^2).\]
From these equations
\[\lambda_2 d(\|\nabla \chi_1\|^2 + \|\nabla \chi_2\|^2) = -(\lambda_1^2 + \lambda_2^2)J_2,\]
\[\lambda_2(1 + d)(\|\phi_1\|^2 + \|\phi_2\|^2) = -d(2\lambda_1 - k)J_2\]
follow. Hence by \(\lambda_2 \neq 0\) we get to
\[d^2(2\lambda_1 - k)(\|\nabla \chi_1\|^2 + \|\nabla \chi_2\|^2) = (1 + d)(\lambda_1^2 + \lambda_2^2)(\|\phi_1\|^2 + \|\phi_2\|^2) \geq 0,\]
which yields \(2\lambda_1 - k \geq 0\). Hence, we obtain the desired assertion. \(\square\)
We define the closed operator $A_0$ of the extension of $-\Delta$ with the Neumann boundary condition in $L^2(\Omega)$. Namely, $A_0$ satisfies

$$A_0v = -\Delta v, \quad v \in \{u \in H^2(\Omega; \mathbb{R}) : \partial u / \partial \nu = 0 \ (x \in \partial \Omega), \int_{\Omega} u \, dx = 0\}.$$ 

and $A_0^s$ stands for the fractional power of $A_0$ with index $s$. We note that $A_0$ is self-adjoint, positive and invertible in $L^2(\Omega)$. Thus $A_0^{-1}$ is defined as a bounded operator in $L^2(\Omega)$.

By using $A_0^{-1}$, we obtain the equation

$$\psi - \langle \psi \rangle = -\frac{d}{1-d} \phi + \frac{\lambda}{1-d} A_0^{-1} \phi,$$  

from (3.2). Inserting this into (3.4) yields

$$\Delta \phi - \alpha g'(w^*) \phi - k \phi + \alpha (k - \lambda) \langle \psi \rangle = -\frac{\lambda}{d} [(1 + d) + (k - \lambda) A_0^{-1}] \phi.$$ 

which is decomposed to

$$\langle g'(w^*) \rangle - (k - \lambda) \langle \psi \rangle = 0, \quad (3.8)$$

and

$$\mathcal{L}(\phi) = \frac{\lambda}{d} ((1 + d) + (k - \lambda) A_0^{-1}) \phi, \quad (3.9)$$

where $\mathcal{L}$ is defined by (1.18). Given a solution $(\phi, \lambda)$ of (3.9), $\langle \psi \rangle$ is determined by (3.8). In the sequel the eigenvalue problem of $\mathcal{A}$ is reduced to that of (3.9).

**Lemma 3.3** The algebraic multiplicity of any eigenvalue $\lambda$ of $\mathcal{A}$ is one if $\text{Re}\lambda < k/2$.

**Proof.** By Lemma 3.2 $\lambda$ is real if $\text{Re}\lambda < k/2$. We let $\lambda_0$ be any eigenvalue with $\lambda_0 < k/2$ and $(\phi_0, \psi_0)$ be a corresponding eigenfunctions.

We first prove that

$$(\mathcal{A} - \lambda_0 I)^2 \left( \begin{array}{c} \phi_1 \\ \psi_1 \end{array} \right) = 0$$

implies $(\phi_1, \psi_1)^T$ being an eigenfunction corresponding to $\lambda_0$, where $I$ is the identity. For the proof, it suffices to show the equation

$$(\mathcal{A} - \lambda_0 I) \left( \begin{array}{c} \phi \\ \psi \end{array} \right) = \left( \begin{array}{c} \phi_0 \\ \psi_0 \end{array} \right)$$

that is,

$$d \Delta \phi + (1 - d) \Delta \psi + \lambda_0 \phi = -\phi_0, \quad (3.10)$$

$$\Delta \psi + g'(w^*) \phi - k \psi + \lambda_0 \psi = -\psi_0 \quad (3.11)$$

14
have no solution.
From (3.10)-(3.11) we see
\[ \Delta \psi = -\frac{1}{1-d}(d\Delta \phi + \lambda_0 \phi + \phi_0), \quad \psi - \langle \psi \rangle = -\frac{d}{1-d} \phi + \frac{1}{1-d}(\lambda_0 A_0^{-1} \phi + A_0^{-1} \phi_0). \]
Thus,
\[ -\Delta \phi + \alpha g'(w^*) \phi + k \phi - \alpha(k - \lambda_0)\langle \psi \rangle \]
\[ -\frac{\lambda_0}{d}[(1 + d) \phi + (k - \lambda_0)A_0^{-1} \phi] \]
\[ = -\alpha \psi_0 + \frac{1}{d} \phi_0 + \frac{k - \lambda_0}{d} A_0^{-1} \phi_0. \]
Decomposing the equation we obtain
\[ \langle g'(w^*) \phi \rangle - (k - \lambda_0)\langle \psi \rangle = -\langle \psi_0 \rangle \]
\[ \mathcal{L}(\phi) - \frac{\lambda_0}{d}[(1 + d) + (k - \lambda_0)A_0^{-1}] \phi \]
\[ = -\alpha (\psi_0 - \langle \psi_0 \rangle) + \frac{1}{d} \phi_0 + \frac{k - \lambda_0}{d} A_0^{-1} \phi_0. \quad (3.12) \]
By
\[ \psi_0 - \langle \psi_0 \rangle = -\frac{d}{1-d} \phi_0 + \frac{\lambda_0}{1-d} A_0^{-1} \phi_0, \]
(3.12) turns to be
\[ \mathcal{L}(\phi) - \frac{\lambda_0}{d}[(1 + d) + (k - \lambda_0)A_0^{-1}] \phi = \frac{1 + d}{d} \phi_0 + \frac{k - 2\lambda_0}{d} A_0^{-1} \phi_0, \quad (3.13) \]
Invoking
\[ \mathcal{L}(\phi_0) - \frac{\lambda_0}{d}[(1 + d) \phi_0 + (k - \lambda_0)A_0^{-1} \phi_0] = 0, \]
the right hand side of (3.13) is orthogonal to \( \phi_0 \) in \( L^2(\Omega) \), namely,
\[ (1 + d)(\phi_0, \phi_0)_{L^2} + (k - 2\lambda_0)(A_0^{-1} \phi_0, \phi_0)_{L^2} \]
\[ = (1 + d)\| \phi_0 \|^2 + (k - 2\lambda_0)\| A_0^{-1/2} \phi_0 \|^2 = 0. \]
This is impossible since \( \phi_0 \neq 0 \) and \( \lambda_0 < k/2 \).
For the general case \( (\mathcal{A} - \lambda_0 I)^m \phi = 0 \) \( (m \geq 2) \), we can prove that it has only the solution \( \phi_0 \) up to multiplication of a constant by the induction argument. Indeed, if there is no solution of \( (\mathcal{A} - \lambda_0 I)^{m-1} \phi_1 = \phi_0 \), then the null space of \( (\mathcal{A} - \lambda_0 I)^m \)
consists of \{c\phi_0 : c \in \mathbb{R}\} by the above argument. This completes the proof. \( \Box \)

As for the equation (3.3), decompose \( \psi = \langle \psi \rangle + \hat{\psi} \). Then the equation is also decomposed as
\[ \langle g'(w^*) \phi \rangle - k\langle \psi \rangle = -\lambda \langle \psi \rangle; \]
\[ \Delta \hat{\psi} + g'(w^*) \phi - \langle g'(w^*) \phi \rangle - k\hat{\psi} = -\lambda \hat{\psi}, \quad \langle \hat{\psi} \rangle = 0. \]
Thus, it suffices to consider the equation for $\hat{\psi}$ since $\langle \psi \rangle$ follows from the solution $(\hat{\psi}, \lambda)$.

Instead of $A$ we may consider the eigenvalue problem of the operator

$$A_1 \left( \begin{array}{c} \phi \\ \psi \end{array} \right) := - \left( d\Delta \phi + (1 - d)\Delta \psi \right),$$

(3.14)

with the domain

$$D(A_1) = \{(\phi, \psi) \in \mathcal{H}_2^2(\Omega) \times \mathcal{H}_2^2(\Omega) : \partial\phi/\partial\nu = \partial\psi/\partial\nu = 0 \ (x \in \partial\Omega)\}.$$

Moreover, from the same computation found in the proof of the above lemma we see that the eigenvalue problem for $A_1$ is reduced to the problem

$$L(\phi) = \lambda \left[ (1 + d) + (k - \lambda)A_0^{-1} \right] \phi,$$

(3.15)

in the domain $D(L)$.

To study (3.15), we introduce a nonlocal eigenvalue problem

$$L[\phi] = \Lambda(d_c + \beta A_0^{-1}) \phi \quad (x \in \Omega), \quad \frac{\partial\phi}{\partial\nu} = 0 \ (x \in \partial\Omega),$$

(3.16)

where $\beta$ is nonnegative parameter. This is an eigenvalue problem of $L$ with the weight in $L^2(\Omega)$. Then we have the following comparison result for the eigenvalues of $L$ and $\Lambda$’s of (3.16), which will be proved in the next section.

**Lemma 3.4** Let $\{\mu_j\}_{j=1,2,\ldots}$ and $\{\Lambda_j(\beta)\}_{j=1,2,\ldots}$ be the sets of eigenvalues of $L$ and (3.16) arranged in increasing order with counting the multiplicity respectively. If $\mu_j > 0$, then

$$\frac{\mu_j}{d_c + \beta/\sigma_2} \leq \Lambda_j(\beta) \leq \frac{\mu_j}{d_c},$$

(3.17)

while if $\mu_j < 0$, then

$$\frac{\mu_j}{d_c} \leq \Lambda_j(\beta) \leq \frac{\mu_j}{d_c + \beta/\sigma_2},$$

(3.18)

holds, where $\sigma_2$ be the second eigenvalue of $-\Delta$ with the Neumann boundary condition in the domain $\Omega$.

Now turn to (3.14) or (3.15). We write the equation as

$$L[\phi] = \lambda \left( \frac{1 + d}{d} + \frac{k - \lambda}{d}A_0^{-1} \right) \phi.$$

Compare this and (3.16) with $d_c = (1 + d)/d$. If there is $\beta^*$ solving the equation

$$\frac{k - \Lambda_j(\beta^*)}{d} = \beta,$$

(3.19)

then $\Lambda_j(\beta^*)$ and the corresponding eigenfunction $\phi_j(\cdot ; \beta^*)$ give solutions to (3.15). Hence, $\Lambda_j(\beta^*)$ is an eigenvalue of $A_1$, that is, $A$. The next lemma is crucial to realize this idea.
Lemma 3.5 Each $\Lambda_j(\beta)$ is continuous in $\beta \geq 0$. Moreover, if $\mu_j > 0$, then $\Lambda_j(\beta)$ is strictly monotone decreasing while if $\mu_j < 0$, then $\Lambda_j(\beta)$ is strictly monotone increasing.

We will prove this key lemma in the next section. We apply Lemmas 3.4 and 3.5 to complete the proof of Theorem 1.3.

Proof of Theorem 1.3: We first see that by Lemma 3.3 the multiplicity of zero eigenvalue for $A$ and $L$ coincides. We put $d_c = (1+d)/d$ and consider the eigenvalue problem (3.16). Assume that

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_p < 0 \leq \mu_{p+1} \leq \cdots \leq \mu_N < k/2.$$  

Invoking $\Lambda_j(0) = \mu_j/d_c$, Lemma 3.5 tells that, for each $j \leq p$, $\Lambda_j(\beta) < 0$ ($\beta > 0$) and $k - \Lambda_j(\beta)$ is monotone decreasing so that a unique solution $\beta = \beta^*_j$ of (3.19) exists. Moreover we see that

$$0 < \beta^*_1 \leq \beta^*_2 \leq \cdots \leq \beta^*_p$$

(see Fig.1), consequently we obtain

$$\Lambda_1(\beta^*_1) \leq \Lambda_2(\beta^*_2) \leq \cdots \Lambda_p(\beta^*_p) < 0.$$  

$$\begin{align*}
\text{(a) } & \mu_j < \mu_i < 0. \\
\text{(b) } & 0 < \mu_j < \mu_i.
\end{align*}$$

Figure 1: The curves $\gamma = \gamma_j(\beta) := (k - \Lambda_j(\beta))/d$ and $\gamma_i(\beta) := (k - \Lambda_i(\beta))/d$ for the case $\mu_j < \mu_i < 0$ in (a) and $0 < \mu_j < \mu_i$ in (b). The horizontal axis is $\beta$-axis. The intersection points at the line $\gamma = \beta$ give $\beta^*_j$ and $\beta^*_i$ respectively. Note $\Lambda_n(0) = \mu_n/d_c$.

Conversely given an eigenvalue $\lambda_m < 0$ of $A$, we consider (3.15), i.e.

$$L[\phi_m] = \lambda_m \left( \frac{1 + d}{d} + \frac{k - \lambda_m}{d} A^{-1}_0 \right) \phi_m$$

(3.20)
Put $\beta_m := (k - \lambda_m)/d$. Then we easily see that $\lambda_m \in \{\Lambda_j(\beta_m)\}_{j=1,2,...}$ the number of the negative eigenvalues of $L$ coincides with that of $A$.

If $0 < \mu_j < k/2$, applying Lemma 3.5 similarly, we see that $\Lambda_j(\beta) > 0$ ($\beta > 0$) and $k - \Lambda_j(\beta)$ is monotone increasing so that there is a unique $\beta_j^* > 0$ solving (3.19). Since $k - \Lambda_j(\beta_j^*) > 0$, $\Lambda_j(\beta_j^*)$ gives a positive eigenvalue of $A$. We also easily see that if $A$ has $q$ positive eigenvalues less than $k/2$, then $L$ has also so.

Finally, (1.21) immediately follows from Lemma 3.4. This completes the proof.

\[\square\]

4 A nonlocal eigenvalue problem

Given a positive $d_c$, we introduce the following inner product:

\[(u,v)_\beta := d_c(u,v)_{L^2} + \beta(A_0^{-1/2}u, A_0^{-1/2}v)_{L^2}\]

and the norm

\[\|u\|_\beta := (d_c\|u\|^2 + \beta\|A_0^{-1/2}u\|^2)^{1/2}\]

for $\beta \geq 0$. We note that

\[\sigma_2\|A_0^{-1/2}u\|^2 \leq \|u\|^2\]

holds, where $\sigma_2$ be the second eigenvalue of $-\Delta$ with the Neumann boundary condition. Thus, the norm $\| \cdot \|_\beta$ gives an equivalent norm to the usual one of $L^2$.

In what follows we fix $d_c$ and we will observe the dependence of eigenvalues on $\beta$. Given a real-valued continuous function $a(x)$ of $\Omega$, we consider the eigenvalue problem

\[
\begin{cases}
L_a[\phi] := -\Delta \phi + a(x)\phi - \langle a(\cdot)\phi \rangle = \Lambda(d_c + \beta A_0^{-1})\phi & (x \in \Omega), \\
\frac{\partial \phi}{\partial \nu} = 0 & (x \in \partial \Omega). 
\end{cases}
\]

Let $\{\Lambda_j(\beta)\}_{j=1,2,...}$ be the set of eigenvalues of (4.1) arranged in increasing order with counting the multiplicity and let $\{\phi_j^\beta(x)\}_{j=1,2,...}$ be the corresponding eigenfunctions. We formulate the variational characterization of the eigenvalues.

Let $\mathcal{M}^n$ be the set of all the $n$-dimensional subspaces of $X$, that is,

\[\mathcal{M}^n = \{X_n \subset X : X_n \text{ is a linear subspace with dim} \ X_n = n\}.\]

Define the Rayleigh quotient as

\[R[\phi; \beta] := \frac{K[\phi]}{\|\phi\|_\beta^2}, \quad K[\phi] := \|\nabla \phi\|^2 + \langle a(\cdot)\phi, \phi \rangle_{L^2}.\]

Then we have

\[\Lambda_n(\beta) = \inf_{X_n \in \mathcal{M}^n} \sup \{R[\phi; \beta] : \phi \in X_n, \phi \neq 0\}.\]
Moreover, by the standard argument we have the alternative formulations as

\[
\Lambda_n(\beta) = \sup_{X_{n-1} \in \mathcal{M}^{n-1}} \inf \{ R[\phi; \beta] : \phi \in X_{n-1}^\perp, \phi \neq 0 \} \tag{4.3}
\]

\[
= \inf \{ R[\phi; \beta] : \phi \in X, \phi \neq 0, \\
(\phi, \phi_j(:; \beta))_\beta = 0 \ (j = 1, 2, \ldots, n-1) \}, \tag{4.4}
\]

where

\[
Y^\perp := \{ u : (u, v)_\beta = 0 \ (\forall v \in Y) \}.
\]

**Lemma 4.1** The zero eigenspace for (4.1) does not depend on \( \beta \geq 0 \).

**Proof.** The zero eigenspace is the set of solutions of \( L_a[\phi] = 0 \) which is independent of \( \beta \geq 0 \). \( \square \)

**Lemma 4.2** For \( 0 < \beta_1 \leq \beta_2 \)

\[
\|\phi\|_{\beta_1} \leq \|\phi\|_{\beta_2}, \quad \|\phi\|_{\beta_2}^2 \leq \frac{\|\phi\|_{\beta_1}^2}{\beta_1} \quad (\phi \in X),
\]

and

\[
\|\phi\|_0^2 \leq \|\phi\|_{\beta_1}^2 \leq \left( 1 + \frac{\beta_1}{\sigma_2 d_e} \right) \|\phi\|_0^2 \quad (\phi \in X)
\]

hold.

**Proof.** The first and the third inequalities immediately follows from the definitions of \( \| \cdot \|_\beta \) and \( \sigma_2 \). For the second inequality, given \( \phi \in X \), we obtain

\[
\frac{\|\phi\|_{\beta_2}^2}{\beta_2} = \frac{d_e}{\beta_2} \|\phi\|^2 + \|A_0^{-1/2}\phi\|^2 \leq \frac{d_e}{\beta_1} \|\phi\|^2 + \|A_0^{-1/2}\phi\|^2 = \frac{\|\phi\|_{\beta_1}^2}{\beta_1}.
\]

This completes the proof. \( \square \)

**Lemma 4.3** If \( 0 < \beta_1 \leq \beta_2 \) and \( \Lambda_n(\beta_2) > 0 \), then \( \Lambda_n(\beta_2) \leq \Lambda_n(\beta_1) \).

**Proof.** From the inf-sup type max-min principle (4.2) and the condition \( \Lambda_n(\beta_2) > 0 \), there exists \( \delta > 0 \) such that

\[
\sup \{ K[\phi]/\|\phi\|_{\beta_2}^2 : \phi \in E, \phi \neq 0 \} \geq \delta \quad (E \in \mathcal{M}^n).
\]

By the assumption we can assert

\[
\sup \{ K[\phi]/\|\phi\|_{\beta_2}^2 : \phi \in E, \phi \neq 0 \} = \sup \{ K[\phi]/\|\phi\|_{\beta_2}^2 : \phi \in E, \phi \neq 0, K[\phi] \geq 0 \}.
\]
From the max-min principle (4.2) we see

$$\Lambda_n(\beta) = \inf_{E \in \mathcal{M}^n} \sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_2}^2} : \phi \in E, \phi \neq 0 \right\}$$

$$= \inf_{E \in \mathcal{M}^n} \sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_2}^2} : \phi \in E, \phi \neq 0, K[\phi] \geq 0 \right\}$$

$$\leq \inf_{E \in \mathcal{M}^n} \sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_1}^2} : \phi \in E, \phi \neq 0, K[\phi] \geq 0 \right\}$$

$$= \inf_{E \in \mathcal{M}^n} \sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_1}^2} : \phi \in E, \phi \neq 0 \right\} = \Lambda_n(\beta_1).$$

This proves the desired inequality.

Lemma 4.4 If $0 < \beta_1 \leq \beta_2$ and $\Lambda_n(\beta_1) > 0$, then $0 < \Lambda_n(\beta_1) \leq (\beta_2/\beta_1)\Lambda_n(\beta_2)$.

Proof. By (4.2) and $\Lambda_n(\beta_1) > 0$ there exists $\delta > 0$ such that

$$\sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_1}^2} : \phi \in E, \phi \neq 0 \right\} \geq \delta \quad (E \in \mathcal{M}^n).$$

Then we see

$$\sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_1}^2} : \phi \in E, \phi \neq 0 \right\} = \sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_1}^2} : \phi \in E, \phi \neq 0, K[\phi] \geq 0 \right\}.$$

Using (4.2) and Lemma 4.2, we have

$$\Lambda_n(\beta_1) = \inf_{E \in \mathcal{M}^n} \sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_1}^2} : \phi \in E, \phi \neq 0 \right\}$$

$$= \inf_{E \in \mathcal{M}^n} \sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_1}^2} : \phi \in E, \phi \neq 0, K[\phi] \geq 0 \right\}$$

$$\leq \inf_{E \in \mathcal{M}^n} \sup \left\{ \frac{\beta_2 K[\phi]}{\beta_1 \|\phi\|_{\beta_2}^2} : \phi \in E, \phi \neq 0, K[\phi] \geq 0 \right\}$$

$$\leq \frac{\beta_2}{\beta_1} \inf_{E \in \mathcal{M}^n} \sup \left\{ \frac{K[\phi]}{\|\phi\|_{\beta_2}^2} : \phi \in E, \phi \neq 0, K[\phi] \geq 0 \right\}$$

$$= \frac{\beta_2}{\beta_1} \Lambda_n(\beta_2).$$

This completes the proof.

Combining Lemmas 4.1, 4.3 and 4.4, we have the following result.
Lemma 4.5 If \( \Lambda_n(\beta_0) > 0 \) for a \( \beta_0 > 0 \), then \( \Lambda_n(\beta) > 0 \) for all \( \beta > 0 \). In addition, \( \Lambda_n(\beta_0) < 0 \) for a \( \beta_0 > 0 \) yields \( \Lambda_n(\beta) < 0 \) for all \( \beta > 0 \).

Proof. The first assertion follows directly from Lemmas 4.3 and 4.4. To see the second assertion, suppose it is not true. Then there is \( \beta_1 > 0 \) such that \( \Lambda_n(\beta_1) \geq 0 \). By Lemma 4.1 \( \Lambda_n(\beta) = 0 \) if \( \Lambda_n(\beta_1) = 0 \), while \( \Lambda(\beta) > 0 \) if \( \Lambda_n(\beta_1) > 0 \), so we get a contradiction.

We have the following inequalities for negative eigenvalues.

Lemma 4.6 Assume that there exists a number \( \beta_0 > 0 \) for which \( \Lambda_n(\beta_0) < 0 \) is met. Then

\[
\Lambda_n(\beta_1) \leq \Lambda_n(\beta_2) \leq (\beta_1/\beta_2)\Lambda_n(\beta_1)
\]

holds if \( 0 < \beta_1 \leq \beta_2 \).

Proof. We notice that the assumption implies \( \Lambda_n(\beta) < 0 \) for any \( \beta > 0 \) by Lemma 4.5. Since \( \Lambda_n(\beta_2) < 0 \), we see that arbitrarily given \( \epsilon \in (0, -\Lambda_n(\beta_2)) \), there exists \( E_\epsilon \in \mathcal{M}^n \) such that

\[
\sup\{K[\phi]/\|\phi\|_{\beta_2}^2 : \phi \in E_\epsilon, \phi \neq 0\} < \Lambda_n(\beta_2) + \epsilon.
\]

Thus \( K[\phi] < 0 \) for \( \phi \in E_\epsilon \). With the aid of this and \( \|\phi\|_{\beta_1}^2 \leq \|\phi\|_{\beta_2}^2 \), we have

\[
\Lambda_n(\beta_1) = \inf_{E \in \mathcal{M}^n} \sup \{K[\phi]/\|\phi\|_{\beta_1}^2 : \phi \in E, \phi \neq 0\} \leq \sup\{K[\phi]/\|\phi\|_{\beta_1}^2 : \phi \in E_\epsilon, \phi \neq 0\} \leq \sup\{K[\phi]/\|\phi\|_{\beta_2}^2 : \phi \in E_\epsilon, \phi \neq 0\} < \Lambda_n(\beta_2) + \epsilon.
\]

Taking \( \epsilon \downarrow 0 \), we obtain \( \Lambda_n(\beta_1) \leq \Lambda_n(\beta_2) \).

We prove the latter inequality. By \( \Lambda_n(\beta_1) < 0 \), for any \( \epsilon \in (0, -\Lambda_n(\beta_1)) \), there exists \( E_\epsilon \in \mathcal{M}^n \) such that

\[
\sup\{K[\phi]/\|\phi\|_{\beta_1}^2 : \phi \in E_\epsilon, \phi \neq 0\} < \Lambda_n(\beta_1) + \epsilon.
\]

Using \( K[\phi] < 0 \) for any \( \phi \in E_\epsilon \) and \( 1/\|\phi\|_{\beta_2}^2 \geq (\beta_1/\beta_2)(1/\|\phi\|_{\beta_1}^2) \), we have

\[
\Lambda_n(\beta_2) = \inf_{E \in \mathcal{M}^n} \sup \{K[\phi]/\|\phi\|_{\beta_2}^2 : \phi \in E, \phi \neq 0\} \leq \sup\{K[\phi]/\|\phi\|_{\beta_2}^2 : \phi \in E_\epsilon, \phi \neq 0\} \leq \sup\left\{\frac{\beta_1 K[\phi]}{\beta_2 \|\phi\|_{\beta_1}^2} : \phi \in E_\epsilon, \phi \neq 0\right\} < \frac{\beta_1}{\beta_2}(\Lambda_n(\beta_1) + \epsilon).
\]

Taking \( \epsilon \downarrow 0 \) yields \( \Lambda_n(\beta_2) \leq (\beta_1/\beta_2)\Lambda_n(\beta_1) \).

Combining the above results leads to the following continuity property for the eigenvalues in \( \beta \).
Proposition 4.7 Every eigenvalue \( \Lambda_n(\beta) \) of (4.1) is continuous in \( \beta > 0 \). More precisely, for each \( \beta_0 > 0 \)

(i) if \( \Lambda_n(\beta_0) < 0 \), then \( \Lambda_n(\beta) < 0 \) for any \( \beta > 0 \) and

\[
\frac{\beta_0}{\beta_0 - |t|} \Lambda_n(\beta_0) \leq \Lambda_n(\beta_0 + t) \leq \frac{\beta_0}{\beta_0 + |t|} \Lambda_n(\beta_0) \quad (t \in (-\beta_0, \beta_0));
\]

(ii) if \( \Lambda_n(\beta_0) > 0 \), then \( \Lambda_n(\beta) > 0 \) for any \( \beta > 0 \) and

\[
\frac{\beta_0}{\beta_0 + |t|} \Lambda_n(\beta_0) \leq \Lambda_n(\beta_0 + t) \leq \frac{\beta_0}{\beta_0 - |t|} \Lambda_n(\beta_0) \quad (t \in (-\beta_0, \beta_0));
\]

(iii) if \( \Lambda_n(\beta_0) = 0 \), then \( \Lambda_n(\beta) = 0 \) for every \( \beta > 0 \).

Proof. The assertion of (iii) follows from Lemma 4.1. We prove (i) by Lemma 4.6. Since \( \Lambda_n(\beta) < 0 \), we have

\[
\frac{\beta_0}{\beta_0 - t} \Lambda_n(\beta_0) \leq \Lambda_n(\beta_0 + t) \leq \frac{\beta_0}{\beta_0 + t} \Lambda_n(\beta_0),
\]

for \( t \geq 0 \). If \( t < 0 \), then

\[
\Lambda_n(\beta_0) \leq \frac{\beta_0 + t}{\beta_0} \Lambda_n(\beta_0 + t), \quad \Lambda_n(\beta_0 + t) \leq \Lambda_n(\beta_0) \leq \frac{\beta_0}{\beta_0 - t} \Lambda_n(\beta_0).
\]

Combining those yields the inequalities of (i).

It is easy to verify (ii) by Lemmas 4.4 and 4.5 similarly. We leave the detail to the leaders.

The next proposition ensures the continuity at \( \beta = 0 \).

Proposition 4.8 For the eigenvalue \( \Lambda_n(\beta) \)

(i) if \( \Lambda_n(0) < 0 \), then

\[
\Lambda_n(0) \leq \Lambda_n(\beta) \leq \frac{\Lambda_n(0)}{1 + (\beta/\sigma_2 d_c)} \quad (\text{for} \quad \beta > 0);
\]

(ii) if \( \Lambda_n(0) > 0 \), then

\[
\frac{\Lambda_n(0)}{1 + (\beta/\sigma_2 d_c)} \leq \Lambda_n(\beta) \leq \Lambda_n(0) \quad (\text{for} \quad \beta > 0);
\]

(iii) if \( \Lambda_n(0) = 0 \), then \( \Lambda_n(\beta) = 0 \) for \( \beta > 0 \).
The proof of this proposition is carried out similarly as done for Proposition 4.7. In fact, Proposition 4.7 follows from Lemmas 4.3, 4.4 and 4.5, which rely on the first and second inequalities of Lemma 4.2. In the proof of Proposition 4.8 the third inequality of Lemma 4.2 works. We leave the detail of the proof to the readers.

Proof of Lemmas 3.4 and 3.5. Put $d_c = (1 + d)/d$. By Proposition 4.7 together with Lemmas 4.4, 4.5 and 4.6, we obtain Lemma 3.5. Invoking $\Lambda_n(0) = \mu_n/d_c = d\mu_n/(1 + d)$ and applying Proposition 4.8 yield Lemma 3.4.

Remark 4.1 We are able to prove that if $\Lambda_n(\beta_0)$ is simple for some $\beta_0$, then there exists $\tau > 0$ such that $\Lambda_n(\beta)$ is differentiable in $\beta$ in $(\beta_0 - \tau, \beta_0 + \tau)$ and

$$
\frac{d\Lambda_n(\beta_0)}{d\beta} = -\Lambda_n(\beta_0)\left(A_0^{-1}\phi_n(\cdot, \beta), \phi_n(\cdot, \beta)\right)
$$

We omit the proof, since it is not used in the proof of the main result of the present article.

5 Lyapunov function for a modified system

In the papers [13] and [12] the authors treat the case $g(u)$, instead of $g(u + v)$, namely

$$
\begin{align*}
\left\{ \begin{array}{ll}
 u_t &= d\Delta u - g(u) + kv, \\
 v_t &= \Delta v + g(u) - kv,
\end{array} \right. \quad (x \in \Omega, \ t > 0)
\end{align*}
$$

(5.1)

Then the system allows the Lyapunov function

$$
\mathcal{E}_0(u, v) := \int_{\Omega} \left\{ \frac{d}{2} |\nabla u|^2 + G(u) + \frac{d}{2} u^2 + \frac{k}{2(1 - d)}(du + v)^2 \right\} \, dx
$$

if $0 < d < 1$, where

$$
G(u) := \int_{\Omega} g(\xi) d\xi.
$$

It is clear that the functional (5.2) differs from (1.5). However, motivated by the fact that both systems possess the Lyapunov functions, we get an idea to consider the following modified system:

$$
\begin{align*}
\left\{ \begin{array}{ll}
 u_t &= d\Delta u - g(u + \gamma v) + kv, \\
 v_t &= \Delta v + g(u + \gamma v) - kv,
\end{array} \right. \quad (x \in \Omega, \ t > 0)
\end{align*}
$$

(5.3)

with the Neumann boundary conditions

$$
\frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0 \quad (x \in \partial\Omega),
$$

(5.4)
where $d$, $k$ are positive constants, $\gamma$ is a constant in $[0, 1]$. This system coincides with (1.1) and (5.1) when $\gamma = 1$ and $\gamma = 0$ respectively.

We show that the system possesses a Lyapunov function.

**Proposition 5.1** Let $0 < d < 1$ and $0 \leq \gamma \leq 1$. Then the following functional gives a Lyapunov function of (5.3) with (5.4):

$$
E_\gamma(u, v) := \int_{\Omega} \left\{ \frac{d}{2} |\nabla (u + \gamma v)|^2 + (1 - d\gamma)G(u + \gamma v) + \frac{dk}{2} (u + \gamma v)^2 + \frac{k(1 - \gamma)}{2(1 - d)} (du + v)^2 + \frac{\gamma}{2} |\nabla (du + v)|^2 \right\} dx,
$$

namely, for a smooth solution $(u(x, t), v(x, t))$

$$
\frac{d}{dt} E_\gamma(u(\cdot, t), v(\cdot, t)) = -\frac{1 - d\gamma^2}{1 - \gamma^2} \int_\Omega |(u + \gamma v)_t|^2 dx - \frac{k(1 - d\gamma)}{1 - d} \int_\Omega |\nabla (du + v)|^2 dx - \frac{\gamma(1 - \gamma)}{1 - d\gamma} \int_\Omega |(du + v)_t|^2 dx \leq 0,
$$

holds.

We note (5.5) coincides with (5.2) and (1.5) for $\gamma = 0$ and $\gamma = 1$ respectively.

**Proof.** We may assume $\gamma < 1$ since the case $\gamma = 1$ is proved in §2. We introduce the new variables

$$
U = u + \gamma v, \quad V = du + v.
$$

By

$$
u = \frac{1}{1 - d\gamma} (U - \gamma V), \quad v = \frac{1}{1 - d\gamma} (-dU + V),
$$

the equations are transformed into

$$
\frac{1 - d\gamma}{1 - \gamma} U_t = d\Delta U + \frac{\gamma(1 - d)}{1 - \gamma} \Delta V - (1 - d\gamma)g(U) - dkU + kV, \quad (5.6)
$$

$$
\frac{1 - d}{1 - d\gamma} U_t + \frac{1 - \gamma}{1 - d\gamma} V_t = \Delta V, \quad (5.7)
$$

and (5.5) is written as

$$
E_\gamma(U, V) = \int_{\Omega} \left\{ \frac{d}{2} |\nabla U|^2 + (1 - d\gamma)G(U) + \frac{dk}{2} U^2 + \frac{k(1 - \gamma)}{2(1 - d)} V^2 + \frac{\gamma}{2} |\nabla V|^2 \right\} dx.
$$
For the solution \((U(x,t), V(x,t))\) of (5.6)-(5.7) with the Neumann conditions

\[
\frac{d}{dt} E_\gamma(U(\cdot,t), V(\cdot,t)) \\
= \int_\Omega \{ d\nabla U \cdot \nabla U_t + (1 - d\gamma)g(U)U_t + kdUU_t + \frac{k(1 - \gamma)}{1 - d}VV_t + \gamma \nabla V \cdot \nabla V_t \} dx \\
= \int_\Omega \{ -d\Delta U + (1 - d\gamma)g(U) + kdU \} U_t + kV \left( \frac{1 - d\gamma}{1 - d} \Delta V - U_t \right) - \gamma V_t \Delta V \} dx \\
= \int_\Omega \left\{ \left[ -\frac{1 - d\gamma}{1 - \gamma} U_t + \frac{\gamma(1 - d)}{1 - \gamma} \Delta V + kV \right] U_t - \frac{k(1 - d\gamma)}{1 - d} |\nabla V|^2 - kVU_t - \gamma V_t \left( \frac{1 - d}{1 - d\gamma} U_t + \frac{1 - \gamma}{1 - d\gamma} V_t \right) \right\} dx \\
= \int_\Omega \left\{ \left[ -\frac{1 - d\gamma}{1 - \gamma} U_t^2 + \frac{\gamma(1 - d)}{1 - \gamma} \left( \frac{1 - d}{1 - d\gamma} U_t + \frac{1 - \gamma}{1 - d\gamma} V_t \right) \right] U_t - \frac{\gamma(1 - d)}{1 - d\gamma} V_t U_t - \frac{\gamma(1 - d)}{1 - d\gamma} V_t^2 - \frac{k(1 - d\gamma)}{1 - d} |\nabla V|^2 \right\} dx \\
= - \int_\Omega \left\{ \frac{1 - d\gamma}{1 - \gamma} \left( 1 - \frac{\gamma(1 - d)^2}{(1 - d\gamma)^2} \right) U_t^2 + \frac{\gamma(1 - d)}{1 - d\gamma} V_t^2 + \frac{k(1 - d\gamma)}{1 - d} |\nabla V|^2 \right\} dx.
\]

Since

\[
1 - \gamma \frac{(1 - d)^2}{(1 - d\gamma)^2} = \frac{(1 - \gamma)(1 - d^2\gamma)}{(1 - d\gamma)^2},
\]

we conclude the proof. \(\square\)

### 6 Appendix: Turing-type instability

As for a specific example

\[
g(w) = kg_0(w), \quad g_0(w) := \frac{w}{(aw + 1)^2}, \quad (6.1)
\]

which is given in [10] and [14], we verify the condition for the instability of a constant steady state of (1.1). Under the condition (1.4), we obtain a constant steady state

\[
(u, v) = (s - g_0(s), g_0(s)). \quad (6.2)
\]

This solution is asymptotically stable for the diffusion-free equations

\[
\frac{du}{dt} = -kg_0(u + v) + kv, \quad \frac{dv}{dt} = kg_0(u + v) - kv \quad (6.3)
\]

under the constraint \(u + v = s\) because the equations are reduced to a scalar equation

\[
\frac{du}{dt} = k(-g_0(s) + s - u).
\]

25
The equilibrium \( u = s - g_0(s) \) is asymptotically stable.

Let \( \sigma_j \) be the \( j \)-th eigenvalue of \( -\Delta \) with the Neumann boundary conditions. The linearized eigenvalue problem

\[
\begin{align*}
- [d\Delta \phi - kg'_0(s)(\phi + \psi) + k\psi] &= \lambda \phi, \\
- [\Delta \psi + kg'_0(s)(\phi + \psi) - k\psi] &= \lambda \psi,
\end{align*}
\]

can be reduced to an infinite sequence of finite eigenvalue problems of

\[
A_j := \begin{pmatrix} d\sigma_j + kg'_0(s) & kg'_0(s) - k \\ -kg'_0(s) & \sigma_j - kg'_0(s) + k \end{pmatrix}.
\]

Hence

\[
\lambda^2 - ((d+1)\sigma_j + k)\lambda + dk\sigma_j[\sigma_j/k + 1 - (1 - 1/d)g'_0(s)] = 0.
\]

Considering

\[
g'_0(w) = \frac{1 - aw}{(aw + 1)^3},
\]

we know that for \( s > 1/a \) there are unstable modes \( j \)'s if \( d \) is sufficiently small.

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