Stability and bifurcation of nonconstant solutions to a reaction-diffusion system with conservation of a mass

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Abstract

We are dealing with a two-component system of reaction-diffusion equations with conservation of a mass. This system is proposed as a conceptual model for cell polarity. Since the system has the conservation of a mass, the steady state problem is reduced to that of a scalar reaction-diffusion equation with a nonlocal term. In particular, we consider the case when the reaction term is linear in one variable. Then the equations are transformed into a similar model of the phase field proposed by Fix and Caginalp, thus the equations allows a Lyapunov function. By investigating the linearized stability of a nonconstant equilibrium solution, we prove that the stability of the equilibrium solution is closely related to that of the nonlocal scalar reaction-diffusion equation. We also exhibit global bifurcation diagrams for equilibrium solutions to specific model equations by numerics together with a normal form near a bifurcation point.
1 Introduction

There are many mathematical models describing pattern formations in nonlinear science. Among other things models of reaction-diffusion equations are very useful for describing the emergence of spatial structures in not only physical and chemical systems but also biological systems. Those have been extensively developed since the discovery of the Turing instability [18] (see also [6], [4], [12], [15], [14], [1] and references therein).

In cellular process we can observe molecular accumulations such as chemotaxis and cell polarity. Recently, Otsuji et al [11] proposed a mass-conserved reaction-diffusion system which exhibits the properties of cell polarity. Ishihara-Otsuji-Mochizuki [10] also show interesting dynamics of solutions to a similar mass-conserved system together with some formal mathematical analysis.

Motivated by their works [10] and [11], we consider the following 2-component reaction-diffusion system:

\[
\begin{align*}
\frac{du}{dt} &= d_1 \Delta u + k_1 f(u, v), \\
\frac{dv}{dt} &= d_2 \Delta v - k_2 f(u, v),
\end{align*}
\]

(1.1)
in a bounded domain, where \(d_j, k_j(j = 1, 2)\) are positive parameters and \(\Delta\) stands for the Laplacian. After rescaling the variables as

\[k_1 t \to t, \quad \sqrt{k_2/d_2} x \to x\]

and putting

\[D := d_1 k_2/d_2 k_1, \quad \tau := k_1/k_2\]

(1.2)
we obtain

\[
\begin{align*}
\frac{du}{dt} &= D \Delta u + f(u, v), \\
\tau \frac{dv}{dt} &= \Delta v - f(u, v) \quad (x \in \Omega),
\end{align*}
\]

(1.3)
where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with smooth boundaries \(\partial \Omega\). Henceforth we consider (1.3) with the Neumann boundary conditions

\[
\frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial v}{\partial \nu} = 0 \quad (x \in \partial \Omega),
\]

(1.4)
or in \(\Omega = [0, L]\) with the periodic boundary conditions

\[
\begin{align*}
u(0, t) &= u(L, t), & u_x(0, t) &= u_x(L, t), \\
v(0, t) &= u(L, t), & v_x(0, t) &= v_x(L, t).
\end{align*}
\]

(1.5)
The aim of the present article is providing a stability theorem for equilibrium solutions to (1.3) together with global bifurcation diagrams for specific functions of \(f\) by numerics.
Throughout the paper, we assume $f$ is sufficiently smooth and the zero set \( \{(u, v) : f(u, v) = 0\} \) is nonempty. As for specific cases, we are interested in
\[
f(u, v) = -\frac{au}{u^2 + b} + v, \quad (1.6)
\]
\[
f(u, v) = -a_1 \left[ \frac{u + v}{a_2(u + v) + 1} \right]^2 - v, \quad (1.7)
\]
and
\[
f(u, v) = a_1(u + v)[(\alpha u + v)(u + v) - a_2], \quad (1.8)
\]
where all the coefficients are positive. The case (1.6) is treated in [10] while (1.7) and (1.8) are in [11].

We observe some basic properties for solutions to (1.3). Firstly, by applying the theorem in Henry[9], the equations (1.3) with (1.4) or (1.5) allows a unique time local solution for the initial conditions
\[
(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) \in X := H^1(\Omega) \times H^1(\Omega), \quad (1.9)
\]
provided that \( \|f(u(\cdot), v(\cdot))\| < \infty \) for \( (u(\cdot), v(\cdot)) \in X \), where \( \| \cdot \| \) stands for the \( L^2 \)-norm. We denote the solution with (1.9) by \( (u(\cdot, t; u_0, v_0), v(\cdot, t; u_0, v_0)) \) for \( t \geq 0 \) generates a \( C^1 \) semiflow in \( X \). Namely,
\[
\{S(t)\}_{t \geq 0} \text{ defined by}
\]
\[
S(t)u_0 := (u(\cdot, t; u_0), v(\cdot, t; u_0)), \quad u_0 := (u_0, v_0) \quad (1.10)
\]
is a \( C^1 \) semiflow in \( X \) (see Appendix in this paper).

Secondly, we notice conservation of a mass such that the solution \( (u, v) = (u(x, t), v(x, t)) \) to (1.3) with the boundary conditions satisfies
\[
\frac{d}{dt} \int_\Omega (u + \tau v)dx = \int_\Omega (D\Delta u + \Delta v)dx = 0,
\]
that is, the quantity
\[
s := \frac{1}{|\Omega|} \int_\Omega (u_0 + \tau v_0)dx \quad (1.11)
\]
is conserved for any \( t \geq 0 \) as long as the solution is defined.

Thirdly, we introduce a new variable
\[
w := Du + v. \quad (1.12)
\]
Then the mass conserved system (1.3) is transformed into
\[
\begin{aligned}
\begin{cases}
    u_t = D\Delta u + f(u, w - Du), \\
    \tau w_t + (1 - \tau D)u_t = \Delta w
\end{cases} \quad (x \in \Omega),
\end{aligned} \quad (1.13)
\]
with
\[ \frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad (x \in \partial \Omega), \tag{1.14} \]
or
\[ \begin{cases} u(0, t) = u(L, t), & u_x(0, t) = u_x(L, t), \\ w(0, t) = w(L, t), & w_x(0, t) = w_x(L, t). \end{cases} \tag{1.15} \]

We note that for the case \( \tau D > 1 \) we can transform (1.3) as
\[ \begin{cases} \tau v_t = \Delta v - f((w - v)/D, v), \\ w_t + (\tau D - 1)v_t = D\Delta w \end{cases} \quad (x \in \Omega), \tag{1.16} \]
which is the same form as (1.13) if interchanging the role of \( u \) and \( v \).

**Remark 1.1** For a specific \( f \) such as
\[ f(u, v) = u - u^3 + kv, \tag{1.17} \]
we notice that (1.13) is a similar to the model equations by Fix [5] and Caginalp [2] if \( \tau_1 D < 1 \). In fact, the phase field model found in [5] and [2] is given as
\[ \begin{cases} \tau_1 \varphi_t = \epsilon^2 \Delta \varphi + W(\varphi) + cT, \\ T_t + \frac{\epsilon}{2} \varphi_t = \kappa \Delta T, \end{cases} \tag{1.18} \]
where \( W(\varphi) = \varphi - \varphi^3 \). Conversely, we can transform (1.18) into (1.1) in what follows. For the case of \( \tau_1 \kappa > \epsilon^2 \), we use
\[ \begin{pmatrix} \varphi \\ T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\epsilon^2 \ell}{2(\tau_1 \kappa - \epsilon^2)} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \]
to obtain
\[ \begin{cases} u_t = \frac{\epsilon^2}{\tau_1} \Delta u + \frac{1}{\tau_1} f(u, v), \\ v_t = \kappa \Delta v - \frac{\epsilon}{2(\tau_1 \kappa - \epsilon^2)} f(u, v), \end{cases} \tag{1.19} \]
where
\[ f(u, v) := u(1 - u^2) + \frac{c \epsilon^2 \ell}{2(\tau_1 \kappa - \epsilon^2)} u + cv. \]

On the other hand, if \( \tau_1 \kappa < \epsilon^2 \), we can use
\[ \begin{pmatrix} \varphi \\ T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\epsilon^2 \ell}{2(\epsilon^2 - \tau_1 \kappa)} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \]
to obtain the similar form.
Fourthly, consider the stationary problem of (1.3)
\[
\begin{cases}
D \Delta u + f(u, v) = 0, \\
\Delta v - f(u, v) = 0
\end{cases}
\quad (x \in \Omega),
\] (1.20)
with the Neumann or the periodic boundary conditions. Let \((u^*(x; s), v^*(x; s))\) be a solution to (1.20) satisfying
\[
s = \frac{1}{|\Omega|} \int_{\Omega} (u^* + \tau v^*) dx.
\] (1.21)
Then \(u^*\) is a solution to
\[
D \Delta u + f \left( u, -Du + s/\tau - (1/\tau - D)|\Omega|^{-1} \int_{\Omega} u \right) = 0,
\] (1.22)
with the boundary conditions. In fact, from (1.20) it follows that
\[
Du^* + v^* = C
\] (1.23)
holds for a constant \(C\). Integrating (1.23) and using (1.21), we easily obtain
\[
v^* = -Du^* + s/\tau - (1/\tau - D)|\Omega|^{-1} \int_{\Omega} u^* dx.
\] (1.24)
Thus, a solution of (1.22) together with (1.24) gives a solution \((u^*, v^*)\) of (1.20), and vice versa.

Now we state mathematical results on the solutions to the mass conserved reaction-diffusion system (1.3). By virtue of the studies by Miyasita [13] and Suzuki-Tasaki [17] for the phase field model (see also [16]), if \(f(u, v) = f_0(u) + kv (k > 0)\), we notice
\[
E(u, v) := \int_{\Omega} \left\{ \frac{D}{2} |\nabla u|^2 - F_0(u) + \frac{kD}{2} u^2 + \frac{\tau k}{2(1-\tau D)} (Du + v)^2 \right\} dx
\] (1.25)
gives a Lyapunov function, where \(F_0(u) = \int f_0(u) du\). More precisely, we have the following lemma:

**Lemma 1.1** Let
\[
f(u, v) = f_0(u) + kv,
\] (1.26)
and \(F_0(u) = \int f_0(u) du\), where \(k\) is positive constant. Assume \(-F_0(u)\) is bounded below and the semiflow \(S(t)\) defined by (1.10) is \(C^1\) in \(X\). If \(0 < \tau D < 1\), then \(E(S(t)u_0)\) is non-increasing in \(t\) and \(E(S(t)u_0) = \text{constant} (\forall t \in \mathbb{R})\) implies that \(u_0\) is an equilibrium. Moreover, the omega limit set of any bounded orbit consists of equilibria, where the bounded orbit implies that \(S(t)u_0\) is uniformly bounded above for \(t \geq 0\).
Next we consider the stability of constant equilibrium solutions. Let \((\bar{u}, \bar{v})\) be a constant equilibrium solution to (1.3) and consider the linearized eigenvalue problem

\[
\begin{cases}
D\Delta \phi + f_u(\bar{u}, \bar{v})\phi + f_v(\bar{u}, \bar{v})\psi = -\lambda \phi, \\
\Delta \psi - f_u(\bar{u}, \bar{v})\phi - f_v(\bar{u}, \bar{v})\psi = -\tau \lambda \psi,
\end{cases}
\]

with the Neumann boundary conditions or the periodic boundary conditions. We let \(\{\sigma_j\}_{j \geq 1}\) be the eigenvalues of \(-\Delta\) with (1.4) or (1.5). Those are labeled in increasing order and counting multiplicity. Let \(\{\varphi_j\}_{j \geq 1}\) be the corresponding normalized eigenfunctions, that is,

\[
\langle \varphi_j, \varphi_m \rangle_{L^2} := \int_{\Omega} \varphi_j(x)\varphi_m(x) dx = \delta_{jm}.
\]

We note that \(\sigma_1 = 0\). The next lemma is due to [10] and [11].

**Lemma 1.2** Assume

\[
0 < f_v(\bar{u}, \bar{v}) < f_u(\bar{u}, \bar{v})/D < f_v(\bar{u}, \bar{v})/\tau D.
\]

Then for \(\sigma_j(j \geq 2)\) satisfying

\[
\sigma_j + f_v(\bar{u}, \bar{v}) - f_u(\bar{u}, \bar{v})/D < 0,
\]

the condition (1.28) allows an eigenvalue \(\lambda = \lambda_j < 0\) of (1.27), where \(\lambda_j\) is obtained by a negative eigenvalue of \(-A_0\)

\[
A_0 := \left( \begin{array}{cc}
-D\sigma_j + f_u(\bar{u}, \bar{v}) & f_v(\bar{u}, \bar{v}) \\
-f_u(\bar{u}, \bar{v})/\tau & -(\sigma_j + f_v(\bar{u}, \bar{v}))/\tau
\end{array} \right).
\]

A corresponding eigenfunction to \(\lambda_j\) is given by

\[
\left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) = \varphi_j(x) \left( \begin{array}{c}
\xi \\
\eta
\end{array} \right),
\]

where \((\xi, \eta)^T\) is an eigenvector of \(-A_0\) to \(\lambda_j\).

We notice that the constant solution is stable under the homogeneous perturbation provided that (1.28) holds. Indeed, for \(\sigma_1 = 0\), the matrix \(-A_0\) has eigenvalues \(\lambda = 0, \lambda = f_v(\bar{u}, \bar{v})/\tau - f_u(\bar{u}, \bar{v}) > 0\). This zero eigenvalue is ruled out if the mass \(s\) is fixed. Therefore, the constant solution is asymptotically stable in the diffusion-free equations under the constraint of the mass. The above lemma implies that the uniform state loses the stability as the diffusion constant \(D\) decreases. Hence, we may say that the Turing instability takes places in this system. We also note that this Turing instability is realized for the specific \(f\) of (1.6), (1.7) and (1.8) if we choose the parameters appropriately.
We consider the next linearized eigenvalue problem for a nonconstant solution \((u^*(\cdot; s), v^*(\cdot; s))\) to (1.20)

\[
\mathcal{L} \begin{pmatrix} \phi \\ \psi \end{pmatrix} := - \begin{pmatrix} D\Delta \phi + f_u(u^*, v^*)\phi + f_v(u^*, v^*)\psi \\ \Delta \psi - f_u(u^*, v^*)\phi - f_v(u^*, v^*)\psi \end{pmatrix} / \tau = \lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{1.31}
\]

with the domain

\[
\mathcal{D}_N(\mathcal{L}) := \mathcal{D}_N \times \mathcal{D}_N, \quad \mathcal{D}_N := \{ \varphi \in H^2(\Omega) : \partial \varphi / \partial \nu = 0 \ (x \in \partial \Omega) \}, \tag{1.32}
\]

or

\[
\mathcal{D}_p(\mathcal{L}) := \mathcal{D}_p \times \mathcal{D}_p, \quad \mathcal{D}_p := \{ \varphi \in H^2(0, L) : \varphi(0) = \varphi(L), \ \varphi_x(0) = \varphi_x(L) = 0 \}. \tag{1.34}
\]

We note that for the periodic case,

\[
\mathcal{L}_0[u^*_x] = 0 \quad \text{in the periodic case.}
\]

Let \(w(x)\) be a solution of

\[
\mathcal{L}_0[\varphi] = k \frac{\tau}{\Omega} \int_{\Omega} \varphi dx = 1.
\]

Corresponding to (1.22), we also introduce the linearized eigenvalue problem

\[
L_0[\varphi] := - \{ D\Delta \varphi + (f_u(u^*, v^*) - Df_v(u^*, v^*))\varphi \\
- (1/\tau - D)f_v(u^*, v^*)\varphi \} = \mu \varphi, \tag{1.36}
\]

with the domain \(\mathcal{D}(L_0) = \mathcal{D}_N\) or \(\mathcal{D}_p\). We see \(L_0[u^*_x] = 0\) in the periodic case.

Let \(w(x)\) be a solution of

\[
L_0[w] = k \frac{\tau}{\Omega} \int_{\Omega} \varphi dx = 1.
\]

with the boundary conditions. The solvability condition for this equation is

\[
1 \in \mathcal{R}(L_0) := \{ g = L_0[\varphi] : \varphi \in \mathcal{D}_K \ (K = N \text{ or } p) \}
\]

If \(L_0\) is invertible, then \(w\) is uniquely determined.

The next theorem is the main result of this article.

**Theorem 1.3** Let \(f_0(u)\) be a \(C^1\) function. Assume (1.26) with \(k > 0\) and \(0 < \tau D < 1\). Let \((u^*(x; s_0), v^*(x; s_0))\) be a non-constant solution to (1.20) with the Neumann boundary conditions or the periodic conditions, where \(s = s_0\) in (1.21). Then all the eigenvalues of \(\mathcal{L}\) of (1.31) in the domain (1.32) or (1.34) are real. Moreover, the following holds:
(i) If every eigenvalue of $L_0$ of (1.36) in $D_N$ is positive, then all the eigenvalues except for zero of $L$ is positive. In addition zero eigenvalue is simple with a corresponding eigenfunction $(\phi, \psi) = (w(x), c_D - Dw(x))$, where

$$c_D := \frac{1}{\tau} - \frac{1 - \tau D}{\tau |\Omega|} \int_{\Omega} wx$$

(1.38)

(ii) If zero eigenvalue of $L_0$ of (1.36) in $D_p$ is simple and the remaining eigenvalues are positive, then all the eigenvalues except for zero of $L$ are positive. In addition the eigenspace of zero eigenvalue is the linear hull of $(u_s^*(\cdot; s_0), v_s^*(\cdot; s_0))$ and $(w(\cdot), c_D - Dw(\cdot))$, where $c_D$ is given by (1.38).

For the result of Theorem 1.3 (i), we can assert that $u^*(\cdot; s)$ is continuously differentiable with $s$ in a neighborhood of $s_0$. As a consequence $w(\cdot) = u_s^*(\cdot; s_0)$ holds.

By virtue of Theorem 1.3 we obtain the following dynamical stability of an equilibrium solution:

**Corollary 1.4** In addition to the assumptions in Theorem 1.3, suppose that the semiflow $S(t)$ defined by (1.10) is $C^1$ in $X = H^1(\Omega) \times H^1(\Omega)$. Let $u^*(\cdot; s_0) := (u^*(\cdot; s_0), v^*(\cdot; s_0))$. Then the following holds:

(i) The case of the Neumann boundary conditions: There exist $\delta > 0$, $\beta > 0$ and $M > 0$ such that if $\tilde{u}_0 = (\tilde{u}_0, \tilde{v}_0) \in X$ satisfies

$$\|\tilde{u}_0 - u^*(\cdot; s_0)\|_X < \delta,$$

then the solution $(u, v) = S(t)\tilde{u}_0$ to (1.3)-(1.4) satisfies

$$\|S(t)\tilde{u}_0 - u^*(\cdot; \tilde{s})\|_X \leq Me^{-\beta t}\|\tilde{u}_0 - u^*(\cdot; \tilde{s})\|_X,$$

where $\tilde{s}$ is given by

$$\tilde{s} = \frac{1}{|\Omega|} \int_{\Omega} (\tilde{u}_0(x) + \tau \tilde{v}_0(x))dx,$$

(1.39)

and $|\tilde{s} - s_0| \to 0$ as $\delta \to 0$.

(ii) The case of the periodic boundary conditions: Let $r_0$ be a positive number. Suppose that for each $s \in I_s := (s_0 - r_0, s_0 + r_0)$, the same condition for $L_0$ as in Theorem 1.3 (ii) holds. Then there exist $\delta_1 > 0$ and $\beta_1 > 0$ such that if

$$\|\tilde{u}_0 - u^*(\cdot; s_0)\|_X < \delta_1,$$

then the solution $(u, v) = S(t)\tilde{u}_0$ to (1.3)-(1.5) satisfies

$$\|S(t)\tilde{u}_0 - u^*(\cdot; + c_1; \tilde{s})\|_X = O(e^{-\beta_1 t})$$

for a positive number $c_1 \in [0, L)$, where $\tilde{s} \in I_s$ is given as (1.39) and $(\tilde{s}, c_1) \to (s_0, 0)$ as $\delta_1 \to 0$. 

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Remark 1.2 A similar result for the dynamical stability to Corollary 1.4 is given in Suzuki-Tasaki [17] by using the Lyapunov function. They prove the Lyapunov stability of the solution. On the other hand by virtue of Theorem 1.3 we can apply the result of [9] to proving the convergence to a solution with a small shift from $s = s_0$.

Remark 1.3 The assertion of Theorem 1.3 implies that if the solution to (1.22) is stable with some nondegeneracy, then the corresponding solution to (1.20) is also stable. However, it does not assure that an unstable solution to (1.22) provides an unstable one to (1.20). However, if $L_0$ in $\mathcal{D}_N$ is invertible, then the zero eigenvalue of $L$ is simple (see Lemma 3.1 in §3).

Remark 1.4 The linearized stability (by (1.36)) of a nonconstant solution to (1.22) is studied in [17] for the Neumann boundary conditions. Due to their result any stable solution in a cylinder domain must be monotone in the axial direction (in the one-dimensional case, the solution is simply monotone). Moreover, if (1.3)-(1.4) with (1.26) allows a unique constant solution, and if it is unstable, we see that the system of equations has a stable invariant set containing a nonconstant solution with the monotonicity. Indeed, the omega limit set of any orbit consists of equilibrium solutions. (See also [17] for nonconstant solutions bifurcating from a constant one of the one-dimension.)

With the aid of some numerics we will give bifurcation diagrams of equilibrium solutions for specific functions $f(u, v)$ in (1.6), (1.7) and (1.8) as the parameter $L$ or $s$ varies. We can observe both supercritical and subcritical bifurcations for the 1-mode solution depending on the parameter values. In addition to the bifurcation of the equilibrium, in the case of (1.8), which is not for (1.26), the numerics tells that a Hopf bifurcation takes place from the 1-mode solution. We also give a mathematical result for a local bifurcation near the bifurcation point at which the constant solution loses the stability (see Theorem 4.1).

Finally we give a remark on a characteristic dynamics of solutions to the model equations reported by [11] and [10]. For convenience of explanation, as in [10], we use the function (1.6) in (1.3) with small $D$ and large $L$ under the periodic boundary conditions. By their numerical simulations, adding a small perturbation to a constant steady state leads to the emergence of a spatial wave pattern (by the Turing instability). It grows until several peaks are created. After the growth of the peaks, some peaks go down. Then the distance of neighboring peaks becomes larger because of the middle ones disappear. If the distance is larger enough, it looks that the pattern settles down to a steady state. However, one peak (or more) collapses abruptly. Simultaneously, adjacent peaks grows more. This process continues until a single peak remains. This is quite an interesting long excursion from the Turing instability to a stable single peak.
We, however, discuss no mathematical treatment for this transient behavior in the present article. That would be a future study.

We organize this paper as follows: In the next section we prove Lemmas 1.1 and 1.2. In §3 we prove Theorem 1.3 together with the Corollary 1.4. In §4 we show numerics for global bifurcation diagrams of equilibrium solutions for specific functions of (1.6), (1.7) and (1.8). Those numerical results are obtained by AUTO. In the same section we also give Theorem 4.1 for a normal form in the center manifold around the bifurcation point where the 1-mode solution emerges. As an Appendix we give a condition on \( f \) that allows the \( C^k \) smoothness of the semiflow \( S(t) \) in \( X \).

2 Proof of Lemmas

As mentioned in the introduction the Lyapunov function (1.25) is found in [13], [16] and [17] where the phase field model by Fix and Caginalp is treated. Combining this and a result in Hale[7] we can prove Lemma 1.1.

Proof of Lemma 1.1: For a solution \( (u(x, t), v(x, t)) \) to (1.3) with (1.4) or (1.5) we compute

\[
\frac{d}{dt} \mathcal{E}(u(\cdot, t), v(\cdot, t)) = \int_{\Omega} [D \nabla u \cdot \nabla u_t - (f_0(u) - kDu)u_t + \frac{\tau k}{1 - \tau D}(Du + v)(Du_t + v_t)] dx
\]

\[
= \int_{\Omega} [(D\Delta u + f_0(u) - kDu)u_t - \frac{\tau k}{1 - \tau D}(Du + v)(Du_t + v_t)] dx
\]

\[
= -\int_{\Omega} [\|u_t\|^2 - k(Du + v)u_t - \frac{\tau k}{1 - \tau D}(Du + v)(Du_t + v_t)] dx
\]

\[
= -\int_{\Omega} \|u_t\|^2 dx - \int_{\Omega} \frac{k}{1 - \tau D} (Du + v)(u_t + \tau v_t) dx
\]

\[
= -\int_{\Omega} \|u_t\|^2 dx - \int_{\Omega} \frac{k}{1 - \tau D} |\nabla(Du + v)|^2 dx.
\]

Hence, for \( \tau D < 1 \) we have

\[
\frac{d}{dt} \mathcal{E}(u(\cdot, t), v(\cdot, t)) \leq 0,
\]

which implies the non-increasing of \( S(t)u_0 \) in \( t \). By the above computation, we also see

\[
\frac{d}{dt} \mathcal{E}(u(\cdot, t), v(\cdot, t)) = 0 \quad (\forall t \in \mathbb{R})
\]

yields

\[
u_t = 0, \quad \nabla(Du + v) = 0 \quad (\forall t \in \mathbb{R}).
\]
Thus we see that $E(u(\cdot,t), v(\cdot,t)) = \text{constant} (\forall t \in \mathbb{R})$ implies $(u(\cdot,t), v(\cdot,t))$ is an equilibrium solution. Considering the assumption on $F_0$, we assert that $E(u, v)$ is bounded below. We see that a bounded orbit $S(t)u_0 (t \geq 0)$ is precompact by the smoothing effect of the reaction-diffusion equations. In the sequel we can apply Lemma 3.8.2 of [7] to obtain that the omega limit set for the solution is included in the set of all the equilibrium solutions. This concludes the proof. 

Lemma 1.2 is proved in [11] and [10]. We give a proof for the reader’s convenience.

Proof of Lemma 1.2: Substituting

$$
\begin{pmatrix}
\phi(x) \\
\psi(x)
\end{pmatrix} = \phi_j(x) \begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
$$

into (1.27) yields

$$
A_0 \begin{pmatrix}
\xi \\
\eta
\end{pmatrix} = -\lambda \begin{pmatrix}
\xi \\
\eta
\end{pmatrix}.
$$

We obtain the equation for $\lambda$ as

$$(\lambda^2 - \{(D + 1/\tau)\sigma_j + f_v/\tau - f_u\} \lambda + \{D\sigma_j^2 + (Df_u - f_u)\sigma_j\})/\tau = 0,$$

where we simply write

$$f_u = f_u(\overline{u}, \overline{v}), \quad f_v = f_v(\overline{u}, \overline{v}).$$

Under the condition (1.28), we have

$$(D + 1/\tau)\sigma_j + f_v/\tau - f_u > 0, \quad Df_v - f_u < 0.$$ 

This fact leads to the desired assertion. 

3 Proof of Theorem 1.3

We first prove the linearized operator $L$ has no imaginary eigenvalues. When $f(u, v) = f_0(u) + kv$, (1.31) turns to be

$$
L \begin{pmatrix}
\phi \\
\psi
\end{pmatrix} = - \left( D\Delta \phi + f_0'(u^*(\cdot; s_0))\phi + k\psi \right) \{\Delta \psi - f_0(u^*(\cdot; s_0))\phi - k\psi\}/\tau = \lambda \begin{pmatrix}
\phi \\
\psi
\end{pmatrix}.
$$

Put

$$a(x) := f_0'(u^*(x; s_0)) - Dk,$$
and
\[ \chi := D\phi + \psi. \]

Then (3.1) is transformed as
\begin{align*}
D\Delta \phi + a(x)\phi + k\chi + \lambda \phi &= 0, \quad (3.2) \\
\Delta \chi + \tau \lambda \chi + \lambda(1 - \tau D)\phi &= 0. \quad (3.3)
\end{align*}

We let
\[ \lambda = \lambda_1 + i\lambda_2, \quad \phi = \phi_1 + i\phi_2, \quad \chi = \chi_1 + i\chi_2. \]

We will prove that \( \lambda_2 \) must be zero. Multiply \( \bar{\phi} \) and \( \bar{\chi} \) with (3.2) and (3.3) respectively. Then integrating them, we have
\begin{align*}
\int_\Omega (-D|\nabla \phi|^2 + a(x)|\phi|^2 + k\chi \bar{\phi} + \lambda|\phi|^2)dx &= 0, \quad (3.4) \\
\int_\Omega (-|\nabla \chi|^2 + \tau \lambda |\chi|^2 + \lambda(1 - \tau D)\phi \bar{\chi})dx &= 0. \quad (3.5)
\end{align*}

Put
\[ \alpha := \int_\Omega (\phi_1 \chi_1 + \phi_2 \chi_2)dx, \quad \beta := \int_\Omega (\phi_1 \chi_2 - \phi_2 \chi_1)dx. \]

Take the imaginary part of (3.4) and write (3.5) the real part and imaginary part separately. Then
\begin{align*}
k\beta + \lambda_2 \|\phi\|^2 &= 0, \quad (3.6) \\
\begin{cases}
\tau \lambda_1 \|\chi\|^2 + (1 - \tau D)(\lambda_1 \alpha + \lambda_2 \beta) = \|\nabla \chi\|^2, \\
\tau \lambda_2 \|\chi\|^2 + (1 - \tau D)(\lambda_2 \alpha - \lambda_1 \beta) = 0,
\end{cases} \quad (3.7)
\end{align*}

where \( \| \cdot \| \) stands for the usual \( L^2 \) norm. If \( \beta = 0 \), then \( \lambda_2 = 0 \). Hence we may assume \( \beta \neq 0 \). From (3.7) we see
\[ \lambda_2 = \frac{(1 - \tau D)\beta \|\nabla \chi\|^2}{(1 - \tau D)\alpha + \tau \|\chi\|^2} + (1 - \tau D)^2 \beta^2. \quad (3.8) \]

Eliminating \( \beta \) from (3.6) and (3.8) yields
\[ \left(1 + \frac{(1 - \tau D)\beta \|\nabla \chi\|^2}{k(1 - \tau D)\alpha + \tau \|\chi\|^2} + (1 - \tau D)^2 \beta^2\right) \lambda_2 = 0. \]

This implies \( \lambda_2 = 0 \), namely, every eigenvalue must be real.

Next, letting \( \lambda_1 \) be the least eigenvalue of \( \mathcal{L} \) in (1.31), we show that the assumption of \( \lambda_1 < 0 \) leads to a contradiction. Define
\[ \mathcal{B} := -\Delta - \tau \lambda_1 \quad (D(\mathcal{B}) = \mathcal{D}_N \text{ or } \mathcal{D}_p) \]

From (3.3)
\[ \chi = \lambda_1(1 - \tau D)\mathcal{B}^{-1}\phi \]

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follows. Hence (3.2)-(3.3) are reduced to
\[ D\Delta \phi + a(x)\phi + \lambda_1(1 - \tau D)k\mathcal{B}[^\phi] = -\lambda_1 \phi \]  
(3.9)

Multiplying \( \phi \) with this equation and integrate by part yields
\[ \int_\Omega \{ D|\nabla \phi|^2 - a(x)|\phi|^2 - \lambda_1(1 - \tau D)k\mathcal{B}[\phi]\phi \} \, dx = \lambda_1 \int_\Omega |\phi|^2 \, dx. \]  
(3.10)

Corresponding to the linearized operator \( L_0 \) defined by (1.36), consider
\[ Q_0[\varphi] := \int_\Omega \{ D|\nabla \varphi|^2 - a(x)|\varphi|^2 \} \, dx + \frac{k(1 - \tau D)}{|\Omega|} \left( \int_\Omega \varphi \, dx \right)^2. \]  
(3.11)

We note that the least eigenvalue \( \mu_1 \) of \( L_0 \) is obtain by
\[ \mu_1 = \inf \{ Q_0[\varphi] : \varphi \in \mathcal{D}(L_0), \| \varphi \| = 1 \}. \]

By the Fourier expansion
\[ \phi(x) = \sum_{j \geq 1} \alpha_j \phi_j(x), \quad \alpha_j = \langle \phi, \phi_j \rangle_{L^2}, \]
we obtain
\[ \mathcal{B}[\phi] = \sum_{j \geq 1} \frac{\alpha_j}{\sigma_j - \tau \lambda_1} \phi_j = -\frac{\alpha_1}{\tau \lambda_1} \phi_1 + \sum_{j \geq 2} \frac{\alpha_j}{\sigma_j - \tau \lambda_1} \phi_j, \]
thus
\[ \langle \mathcal{B}[\phi], \phi \rangle_{L^2} = -\frac{\alpha_1^2}{\tau \lambda_1} + \sum_{j \geq 2} \frac{\alpha_j^2}{\sigma_j - \tau \lambda_1}. \]  
(3.12)

Using (3.10), (3.11) and (3.12) together with
\[ \alpha_1 = \frac{1}{|\Omega|^{1/2}} \int_\Omega \phi \, dx, \]
we obtain
\[ Q_0[\phi] - k(1 - \tau D)\lambda_1 \sum_{j \geq 2} \frac{\alpha_j^2}{\sigma_j - \tau \lambda_1} = \lambda_1 \| \phi \|^2 \]  
(3.13)

The assumption of Theorem 1.3 implies that \( \mu_1 \geq 0 \). We thereby obtain
\[ Q_0[\phi] \geq 0. \]

Consequently, by the assumption \( \lambda_1 < 0 \), we get to
\[ -k(1 - \tau D) \sum_{j \geq 2} \frac{\alpha_j^2}{\sigma_j - \tau \lambda_1} \geq \| \phi \|^2 \]
which yields $\|\phi\| = 0$. This is a contradiction. Hence, $\lambda$ must be nonnegative.

Now we consider the case $\lambda_1 = 0$. Then (3.2) and (3.3) imply

$$D\Delta \phi + a(x)\phi + k\chi = 0, \quad \Delta \chi = 0. \quad (3.14)$$

Let $\chi(x) = c$. Then

$$c = \frac{D}{|\Omega|} \int_\Omega \phi dx + \frac{1}{|\Omega|} \int_\Omega \psi dx.$$

Put

$$\rho = \frac{1}{|\Omega|} \int_\Omega (\phi + \tau \psi) dx.$$

From these two equations it follows that

$$c = \frac{\rho}{\tau} - \frac{1 - \tau D}{\tau |\Omega|} \int_\Omega \phi dx.$$

Thus (3.14) are reduced to

$$L_0[\phi] = D\Delta \phi + a(x)\phi - \frac{(1 - \tau D)k}{\tau |\Omega|} \int_\Omega \phi dx = -\frac{\rho}{\tau}. \quad (3.15)$$

By the assumption of the theorem we can assert that only $\phi = \rho w(\cdot)$ is a solution of (3.15) for the Neumann case while all the solutions for the periodic case are given by $\phi = cu^*_x(\cdot; s_0) + \rho w(\cdot)$.

In order to complete the proof we need to show that there is no general eigenfunction of zero eigenvalue of $L$. We have the following lemma:

**Lemma 3.1** Under the same assumptions of Theorem 1.3 the generalized eigenspace of zero eigenvalue of $L$ consists of $c(w(x), c_D - Dw(x))^T$ (c is real number) for the Neumann case, while it consists of the linear combination of $(w(x), c_D - Dw(x))^T$ and $(u^*_x(\cdot; s_0), v^*_x(\cdot; s_0))^T$ for the periodic cases, where $c_D$ is given by (1.38).

**Proof.** We consider the periodic case. Let

$$L \begin{pmatrix} \phi \\ \psi \end{pmatrix} = c_0 \begin{pmatrix} w(\cdot) \\ c_D - Dw(\cdot) \end{pmatrix} + c_1 \begin{pmatrix} u^*_x(\cdot; s_0) \\ v^*_x(\cdot; s_0) \end{pmatrix}, \quad (3.16)$$

where $c_0$ and $c_1$ are real numbers. We note that by the assumption the null space of $L$ in $D_p \times D_p$ consists of the linear hull of $(w(\cdot), c_D - Dw(\cdot))^T$ and $(u^*_x(\cdot; s_0), v^*_x(\cdot; s_0))^T$. We prove that there is no solution for any $(c_0, c_1) \neq (0,0)$. This implies the assertion of the lemma.

Define the adjoint operator

$$L^* \begin{pmatrix} p \\ q \end{pmatrix} := - \begin{pmatrix} Dp_{xx} + f_0'(u^*(\cdot; s_0))p - (1/\tau)f_0'(u^*(\cdot; s_0))q \\ (1/\tau)q_{xx} + kp - (k/\tau)q \end{pmatrix}. \quad (3.17)$$
We solve
\[ \mathcal{L}^* \left( \begin{array}{c} p \\ q \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \]
that is,
\[ Dp_{xx} + f_0'(u^*)(p - q/\tau) = 0, \quad (1/\tau)q_{xx} + k(p - q/\tau) = 0. \quad (3.18) \]
These equations are reduced to
\[ D(p - q/\tau)_{xx} + (f_0'(u^*) - Dk)(p - q/\tau) = 0. \]
By the assumption of the simplicity of zero eigenvalue of \( L_0 \), we solve this equation as
\[ p - q/\tau = \alpha u^*_x \quad (\alpha \in \mathbb{R}). \]
Applying this to the second equation of (3.18) yields
\[ q_{xx} + \tau k \alpha u^*_x = 0. \]
After a simple integration, we thereby obtain
\[ \begin{cases} p = \alpha u^*_x + \alpha k \left( - \int_0^x u^* dx + \frac{x}{L} \int_0^L u^* dx \right) + \beta/\tau, \\ q = \alpha k \tau \left( - \int_0^x u^* dx + \frac{x}{L} \int_0^L u^* dx \right) + \beta \end{cases} \quad (\alpha, \beta \in \mathbb{R}). \quad (3.19) \]
(\( \alpha, \beta \in \mathbb{R} \)). On the other hand the equation
\[ \mathcal{L}_0 \left( \begin{array}{c} \phi \\ \psi \end{array} \right) = c_0 \left( \begin{array}{c} w \\ c_D - Dw \end{array} \right), \quad (3.20) \]
is solvable for nonzero \( c_0 \) if and only if
\[ \int_0^L \{ wp + (c_D - Dw)q \} dx = 0 \quad (\forall (p, q) \in \mathcal{N}(\mathcal{L}^*)). \quad (3.21) \]
However, we see that for \((p, q) = (1/\tau, 1)\)
\[ \frac{1}{L} \int_0^L (w/\tau + c_D - Dw) dx = 1/\tau \neq 0. \]
This implies that there is no solution to (3.20) for \( c_0 \neq 0 \).
Next consider
\[ \mathcal{L} \left( \begin{array}{c} \phi \\ \psi \end{array} \right) = c_1 \left( \begin{array}{c} u^*_x \\ v^*_x \end{array} \right), \quad (3.22) \]
which is solvable for \( c_1 \neq 0 \) if and only if
\[ \int_0^L (u^*_x p + v^*_x q) dx = 0 \quad (\forall (p, q) \in \mathcal{N}(\mathcal{L}^*)). \quad (3.23) \]
Put
\[ Y(x) := -\int_0^x u^* dx + \frac{x}{L} \int_0^L u^* dx. \]

For \((p, q)\) of (3.19) we easily verify
\[
\int_0^L (u_*^* + v_*^*) dx \\
= \alpha \int_0^L \left\{ (u_*^*)^2 + k(u_*^* + \tau v_*^*) Y \right\} dx \\
= \alpha \left\{ \int_0^L (u_*^*)^2 dx - \int_0^L k(u_*^* + \tau v_*^*) Y dx \right\}. \tag{3.24}
\]

We compute as
\[
\int_0^L (u_*^* + \tau v_*^*) Y_* dx \\
= \int_0^L (u_*^* + \tau v_*^*) \left\{ -u_*^* + \left( \frac{1}{L} \right) \int_0^L u_*^* dx \right\} dx \\
= -\int_0^L ((u_*^*)^2 + \tau u_*^* v_*^*) dx + s \int_0^L u_*^* dx, \tag{3.25}
\]
and
\[
\int_0^L ((u_*^*)^2 + \tau u_*^* v_*^*) dx \\
= \int_0^L \left\{ (u_*^*)^2 + \tau u_*^* \left( -Du_*^* + \frac{1}{\tau} \left( s - \frac{1 - \tau D}{L} \int_0^L u_*^* dx \right) \right) \right\} dx \\
= \int_0^L \left\{ (1 - \tau D)(u_*^*)^2 + u_*^* \left( s - \frac{1 - \tau D}{L} \int_0^L u_*^* dx \right) \right\} dx \\
= (1 - \tau D) \int_0^L (u_*^*)^2 dx + s \int_0^L u_*^* dx - \frac{1 - \tau D}{L} \left( \int_0^L u_*^* dx \right)^2. \tag{3.26}
\]
Applying (3.26) to (3.25) yields
\[
\int_0^L (u_*^* + \tau v_*^*) Y_* dx = (1 - \tau D) \left\{ -\int_0^L (u_*^*)^2 dx + \frac{1}{L} \left( \int_0^L u_*^* dx \right)^2 \right\}. \tag{3.27}
\]
Using
\[
\left( \int_0^L u_*^* dx \right)^2 \leq L \int_0^L (u_*^*)^2 dx,
\]
we obtain
\[
\int_0^L (u_*^* + \tau v_*^*) Y_* dx \leq 0
\]
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In the sequel, the left hand side of (3.24) must be positive. This implies that there is no solution of (3.22) for \( c_1 \neq 0 \).

Since it is easier to prove the assertion for the Neumann case, we left the detail to the readers.

By Lemma 3.1 we can complete the proof of Theorem 1.3.

**Proof of Corollary 1.4:** First consider the Neumann boundary case. Let

\[
G(u, s) := D\Delta u + f \left( u, -Du + s/\tau - (1/\tau - D)|\Omega|^{-1} \int_{\Omega} u dx \right), \tag{3.28}
\]

which is a mapping

\[
G(\cdot, \cdot) : C^{2+\alpha}(\Omega) \times (s_0 - \delta, s_0 + \delta) \rightarrow C^\alpha(\Omega),
\]

where \( 0 < \alpha < 1 \) (\( n \geq 2 \)), \( \alpha = 0 \) (\( n = 1 \)). We note that \( G \) is of class \( C^1 \). By the assumption, we have \( G(u^*, s_0) = 0 \) and the invertibility of \( (\partial G/\partial u)(u^*, s_0) \). We can apply the implicit function theorem to obtain a solution \( u = u^*(\cdot; s) \) satisfying

\[
G(u^*(\cdot; s), s) = 0, \quad |s - s_0| < \eta,
\]

where \( \eta \) is a small number. Then \( u^*(\cdot; s) \) is \( C^1 \) with respect to \( s \) near \( s = s_0 \). If \( ||\tilde{u}_0 - u^*(\cdot; s_0)|| \) is small enough, then \( |s - s_0| < \eta \). By the assumption on \( L_0 \), we see that \( L \) has no zero eigenvalue if we consider it in the space

\[
\{ (\phi, \psi) \in \mathcal{D}(L) : \int_{\Omega} \phi dx + \int_{\Omega} \tau \psi dx = 0 \}.
\]

By virtue of the conservation of the mass, given \( \tilde{u}_0 \) with

\[
s = \frac{1}{|\Omega|} \int_{\Omega} (\tilde{u}_0 + \tau \tilde{v}_0) dx, \quad |s - s_0| < \eta,
\]

we put \( v(\cdot, t) := u(\cdot, t; \tilde{u}_0) - u^*(\cdot; s) \). Then we may consider \( v(\cdot, t) \) in the space

\[
X_s := \{ u \in X : \int_{\Omega} (u + \tau v) dx = 0 \}.
\]

Hence, Theorem 5.1.1 of [9] is applicable. We obtain the assertion for the Neumann case.

Next consider the periodic case. As seen above, given \( \tilde{u}_0 \), we can restrict \( v(\cdot, t) := u(\cdot, t; \tilde{u}_0) - u^*(\cdot; s) \) in the space \( X_s \) if \( s \in I_s \). This implies that the zero eigenvalue of \( L \) corresponding to \( (w(\cdot), c_D - Dw(\cdot)) \) is ruled out when the flow is considered in \( X_s \). Since the map \( c \mapsto u(\cdot + c; s) \) creates a \( C^2 \) curve in \( X_s \), we can apply the result of [9] (Chap. 5, Exc. 6) to obtain the desired assertion. This concludes the proof. 

\[\square\]
4 Bifurcation diagrams

In this section we study the bifurcation structure of stationary solutions to (1.3) which bifurcate from the uniform state under the assumption (1.28), that is, the Turing unstable case. It should be noticed here that solutions to the Neumann problem can be considered as those of the periodic boundary problem with the period double size of the original domain. More precisely, suppose \((u(x, t), v(x, t))\) is a solution to (1.3) in \(\Omega = [0, L]\) with the Neumann boundary conditions (1.4), and we extend the functions \(u\) and \(v\) for \(x \in [0, 2L]\) as follows:

\[
\begin{align*}
   u(x, t) &= \begin{cases} 
     u(x, t), & x \in [0, L], \\
     u(2L - x, t), & x \in (L, 2L], 
   \end{cases} \\
   v(x, t) &= \begin{cases} 
     v(x, t), & x \in [0, L], \\
     v(2L - x, t), & x \in (L, 2L].
   \end{cases}
\end{align*}
\]

(4.1)

It is easy to check that these extended functions solve (1.3) in \(\Omega = [0, 2L]\) with the periodic boundary conditions of period 2L. On the contrary, suppose that a pair of 2L-periodic functions \((u(x, t), v(x, t))\) is a solution to (1.3) and that they have even symmetry: \(u(-x, t) = u(x, t)\) and \(v(-x, t) = v(x, t)\). Then it solves (1.3) - (1.4). Therefore, we only consider the bifurcation structure of the stationary solutions with the periodic boundary conditions and even symmetry throughout this section.

Let \((\overline{u}, \overline{v})\) be the constant equilibrium solution, that is, \(f(\overline{u}, \overline{v}) = 0\) and \(\overline{u} + \tau \overline{v} = s\). We study a local bifurcation about this constant equilibrium. By noticing that \(u\) and \(v\) are even and 2L-periodic, the Fourier expansions of them about the equilibrium can be described as

\[
\begin{align*}
   u(x, t) - \overline{u} &= \alpha_0 + 2 \sum_{m > 0} \alpha_m \cos mk_0 x, \\
   v(x, t) - \overline{v} &= \beta_0 + 2 \sum_{m > 0} \beta_m \cos mk_0 x.
\end{align*}
\]

(4.2)

Here,

\[k_0 := \pi / L,\]

and both \(\alpha_m\) and \(\beta_m\) are real valued functions of \(t\) for \(m \geq 0\). In addition to (4.2) we can also express the Fourier expansions as follows:

\[
\begin{align*}
   u(x, t) - \overline{u} &= \sum_{m \in \mathbb{Z}} \alpha_m e^{imk_0 x}, \\
   v(x, t) - \overline{v} &= \sum_{m \in \mathbb{Z}} \beta_m e^{imk_0 x},
\end{align*}
\]

(4.3)

where, \(\alpha_{-m} = \alpha_m\) and \(\beta_{-m} = \beta_m\).
The equation (1.3) now takes the following decomposition in terms of the Fourier modes:
\[
\begin{align*}
\dot{\alpha}_m &= -m^2 k_0^2 D \alpha_m + \hat{f}_m \quad (m \geq 0), \\
\tau \dot{\beta}_m &= -m^2 k_0^2 \beta_m - \hat{f}_m \quad (m \geq 0).
\end{align*}
\] (4.4)

Here, \( \hat{f}_m \) denotes the \( m \)-th Fourier coefficient of the function \( f(u, v) \). Notice that \( \alpha_0 + \tau \beta_0 = 0 \) since it is equivalent to find solutions to (1.3) with the constraint \( s = \int_\Omega (u + \tau v) dx/|\Omega| \). In sequel the equations we have to solve are the following:
\[
\begin{align*}
\dot{\alpha}_m &= -m^2 k_0^2 D \alpha_m + \hat{f}_m \quad (m \geq 0), \\
\tau \dot{\beta}_m &= -m^2 k_0^2 \beta_m - \hat{f}_m \quad (m > 0), \\
\tau \beta_0 &= -\alpha_0.
\end{align*}
\] (4.5)

By virtue of the constraint \( \alpha_0 + \tau \beta_0 = 0 \) we can remove the degeneracy of the stationary problem and calculate the continuation of the stationary solution for a given constant \( s \).

To determine a local bifurcation equation about the constant equilibrium, we take the Taylor expansion of \( f \) at \((\bar{u}, \bar{v})\):
\[
\begin{align*}
f(u, v) &= f(\bar{u}, \bar{v}) + f_u \cdot (u - \bar{u}) + f_v \cdot (v - \bar{v}) \\
&\quad + \frac{f_{uu}}{2} (u - \bar{u})^2 + f_{uv} \cdot (u - \bar{u})(v - \bar{v}) + \frac{f_{vv}}{2} (v - \bar{v})^2 \\
&\quad + \frac{f_{uuu}}{6} (u - \bar{u})^3 + \cdots + O(4),
\end{align*}
\] (4.6)

where each coefficient is a corresponding partial derivative of \( f \) evaluated at \((\bar{u}, \bar{v})\). Then the zero’th order equation of (4.5) can be described as
\[
\begin{align*}
\dot{\alpha}_0 &= f_u \cdot \alpha_0 + f_v \cdot \beta_0 + O(2) \\
&= \left( f_u - \frac{f_v}{\tau} \right) \alpha_0 + O(2). \tag{4.7}
\end{align*}
\]

We know that the equilibrium \( \alpha_0 = 0 \) is stable by the assumption (1.28).

The linearized matrix for the non-zero mode \( m \) of (4.5) is given by
\[
A_m = \begin{pmatrix}
-m^2 k_0^2 D + f_u & \frac{f_v}{\tau} \\
\frac{f_u}{\tau} & -m^2 k_0^2 + f_v
\end{pmatrix}.
\]

Since trace \( A_m = -(1 + \tau D) m^2 k_0^2 /\tau + f_u - f_v /\tau \) from (1.28), the \( m \)’th mode is unstable if and only if \( \det A_m < 0 \). Now the expression
\[
\det A_m = \frac{m^4 k_0^4 D - (f_u - D f_v) m^2 k_0^2}{\tau}
\]
shows that \( \det A_m \) can be negative for \( k := mk_0 \in (0, \sqrt{(f_u - Df_v)/D}) \) and a bifurcation to a nontrivial solution takes place at \( k = \sqrt{(f_u - Df_v)/D} \). Here we used the Turing unstable assumption \( f_u - Df_v > 0 \) in (1.28).

Let us take \( k_0 \) as a bifurcation parameter and search bifurcated branches numerically. In addition to the parameter \( k_0 \) bifurcation diagrams for \( s \) are also exhibited in the successive subsections. We also show the local bifurcation analysis corresponding to the numerical results in the last subsection.

4.1 Numerical results for (1.6)

We introduce numerically obtained bifurcation diagrams of stationary solutions to (1.3) for the specific cases \( f \): (1.6). We solve (4.5) numerically by cutting off the Fourier modes up to 64 modes. All the simulations are calculated by AUTO, a software package for bifurcation analysis ([3]).

We first consider \( k_0 \) as a bifurcation parameter and set the other constants as follows:

\[
\begin{align*}
s &= 5 \\
D &= 0.01 \\
a &= 1 \\
b &= 0.1 \\
\tau &= 1
\end{align*}
\]

We show a branch of stationary solutions for the first mode by taking \( k_0 \) as the horizontal axis in Figure 1. It turns out that a stationary solution bifurcates from the uniform state in subcritical pitchfork, therefore, both of the nontrivial stationary solution and the trivial state can be stable simultaneously for an appropriate \( k_0 \). It should be remarked that any branch of higher mode solutions can be obtained by a simple scaling of the first mode since \( k_0 = \pi/L \).

We also show a bifurcation diagram for the bifurcation parameter \( s \) and for \( k_0 = 2 \) in Figure 1. Here, the branch of the uniform steady state forms a S-shaped curve and both ends of the branch of nontrivial stationary solutions connect to the S-shaped curve in subcritical pitchfork.

4.2 Numerical results for (1.7)

In similar to to the previous subsection we introduce the numerically obtained bifurcation diagrams of the stationary solutions to (1.3) for the specific case of \( f \): (1.7).

We first consider \( k_0 \) as a bifurcation parameter and set the other constants as follows:

\[
\begin{align*}
D &= 0.01 \\
\alpha_1 &= 2.5 \\
\alpha_2 &= 0.7 \\
\tau &= 1
\end{align*}
\]
Bifurcation diagrams for different values of $s$ are shown in Figure 2. It turns out that the direction of pitchfork bifurcations from the uniform state depends on $s$. As a result both of the nontrivial stationary solution and the trivial state can be stable simultaneously by taking an appropriate $k_0$ and $s$.

### 4.3 Numerical results for (1.8)

In similar to the previous subsections, we show the numerically obtained bifurcation diagrams of the stationary solutions to (1.3) for the specific case of $f$: (1.8).

We first consider $k_0$ as a bifurcation parameter and the other constants are given as follows:

\[
\begin{align*}
    s &= 10 \quad D = 0.1 \quad \tau = 1 \\
    \alpha &= 0.1 \quad a_1 = 0.5 \quad a_2 = 2.2
\end{align*}
\]

We show branches of stationary solutions for several modes by taking $k_0$ as the horizontal axis in Figure 3.

Next, we show, in Figure 4, another continuation from the previous 1-mode stationary solution at $k_0 = 1$ by taking $s \in [0.1, 10]$ as the bifurcation parameter.
Figure 2: Bifurcation diagrams of the stationary solutions for (1.7) for $s = 3.0, 5.0, 10.0$, respectively. The horizontal axis denotes the bifurcation parameter $k_0$ and the vertical axis denotes $L^2$ norm of the solution. We also show a bifurcation diagram for $s$ in the bottom right.

Figure 3: Bifurcation diagram of the stationary solutions for (1.8). The horizontal axis denotes the bifurcation parameter $k_0$ and the vertical axis denotes $L^2$ norm of the solution. The bifurcating branches for $m = 1, 2, 3, 4, 5$ modes from the right are observed.

instead of $k_0$. Then we find a Hopf bifurcation on this branch. In fact we can numerically observe periodic motions by solving (1.3) with small initial data.
4.4 Bifurcation analysis

As mentioned earlier, the bifurcation to the nontrivial solution from the uniform state takes place when \( k = mk_0 = \sqrt{(f_u - Df_v)/\bar{D}} \). In this subsection let us focus on the bifurcation to the first branch, namely, \( m = 1 \). The purpose of this sub-section is to derive the normal form of the local bifurcation to the first branch at \( k_0 = k^* := \sqrt{(f_u - Df_v)/\bar{D}} \). The matrix \( A_1 \) has 0 eigenvalue if and only if \( k_0 = k^* \). In order to calculate the critical mode explicitly we take a new coordinate \((\eta, \theta)\) instead of \((\alpha_1, \beta_1)\) so that \( \eta \) corresponds to the critical mode. More precisely, when \( k_0 = k^* \) the eigenvalues of \( A_1 \) are 0 and trace\( A_1 = Df_v - f_u \). The linearized matrix \( A_1 \) for 1-mode can be diagonalized

\[
T^{-1}A_1T = \begin{pmatrix} 0 & 0 \\ 0 & \text{trace}A_1 \end{pmatrix}
\]

by

\[
T = \begin{pmatrix} 1 & -f_u \\ -D & \frac{f_u}{D\tau} \end{pmatrix}.
\]

The new coordinate \((\eta, \theta)\) for 1-mode is defined by

\[
\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = T \begin{pmatrix} \eta \\ \theta \end{pmatrix} = \begin{pmatrix} \eta - f_u\theta \\ -D\eta + \frac{f_u}{D\tau}\theta \end{pmatrix}.
\]

We are interested in a small \( \delta \)-neighborhood of 0 for the critical mode \( \eta \). All the other modes are slave mode in the sense that those are of higher orders in the center manifold reduction. In sequel we can conclude

\[
\alpha_1 = \eta + O(\delta^2), \quad \beta_1 = -D\eta + O(\delta^2).
\]

Figure 4: The left figure shows a branch of the stationary solutions for (1.8). The horizontal axis denotes the bifurcation parameter \( s \) and the vertical axis denotes \( L^2 \) norm of the solution. The black square indicates a Hopf bifurcation point. The scaled profiles of the corresponding periodic solutions \( u \) on the branch are shown in the right figure.
By plugging the Fourier expansion into (4.6) we obtain the equation for 1-mode as follows:
\[
\left( \begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) = A_1 \left( \begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) + N \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{3} \end{array} \right) + O(\delta^4),
\]
where the quadratic and cubic nonlinear terms $N$ of $\hat{f}_1$ are given by
\[
N = \frac{f_{uu}}{2} \sum_{m_1+m_2=1} \alpha_{m_1} \alpha_{m_2} + f_{uw} \sum_{m_1+m_2=1} \alpha_{m_1} \beta_{m_2} + \frac{f_{uv}}{2} \sum_{m_1+m_2=1} \beta_{m_1} \beta_{m_2}
+ \frac{f_{uuu}}{6} \sum_{m_1+m_2+m_3=1} \alpha_{m_1} \alpha_{m_2} \alpha_{m_3} + \frac{f_{uw}}{2} \sum_{m_1+m_2+m_3=1} \alpha_{m_1} \alpha_{m_2} \beta_{m_3}
+ \frac{f_{uvw}}{2} \sum_{m_1+m_2+m_3=1} \alpha_{m_1} \beta_{m_2} \beta_{m_3} + \frac{f_{vvv}}{6} \sum_{m_1+m_2+m_3=1} \beta_{m_1} \beta_{m_2} \beta_{m_3}
\]

If $m_1 \neq \pm 1$ and $m_2 \neq \pm 1$ in the above quadratic terms, then the corresponding quadratic term such as $\alpha_{m_1} \alpha_{m_2}$ is $O(\delta^4)$ since both $\alpha_{m_1}$ and $\alpha_{m_2}$ are slave modes. Therefore, from the above quadratic sums, $O(\delta^3)$ terms appear only when one of $m_j = \pm 1$ ($j = 1, 2$). It turns out that the possible pairs $(m_1, m_2)$ are $(1, 0), (0, 1)$ and $(-1, 2), (2, -1)$. Thus we need to take into account the cooperate modes, that is, $m = 0, 2$. Moreover, if one of $m_j (j = 1, 2, 3)$ is a slave mode then the corresponding cubic term such as $\alpha_{m_1} \alpha_{m_2} \alpha_{m_3}$ is $O(\delta^4)$. Therefore, from the above cubic sums, the $O(\delta^3)$ terms appear only when $m_j = \pm 1$ for $j = 1, 2, 3$. Thus the possible triples $(m_1, m_2, m_3)$ are $(1, 1, -1), (1, -1, 1), (-1, 1, 1)$. We then calculate the $O(\delta^3)$ terms of $N$ as follows:
\[
N = f_{uu}(\alpha_1 \alpha_0 + \alpha_1 \alpha_2) + f_{uv}(\alpha_1 \beta_0 + \alpha_1 \beta_2 + \alpha_0 \beta_1 + \alpha_2 \beta_1) + f_{uv}(\beta_1 \beta_0 + \beta_1 \beta_2)
+ \frac{f_{uuu}}{2} \alpha_1^3 + \frac{3f_{uvw}}{2} \alpha_1^2 \beta_1 + \frac{3f_{uvw}}{2} \alpha_1 \beta_1^2 + \frac{f_{vvv}}{2} \beta_1^3 + O(\delta^4).
\]
Here we used the fact that $\alpha_{-1} = \alpha_1$ and $\beta_{-1} = \beta_1$. By plugging (4.8) and the constraint $\beta_0 = -\alpha_0/\tau$ into the above we further obtain
\[
N = \{f_{uu} - DF_{uv} - (f_{uv} - Df_{vv})/\tau\} \eta_0
+ (f_{uu} - Df_{uv}) \eta_0 \alpha_0 + (f_{uv} - Df_{vv}) \eta_0 \beta_0
+ \frac{1}{2} (f_{uuu} - 3f_{uuu} D + 3f_{uvv} D^2 - f_{vvv} D^3) \eta^3 + O(\delta^4).
\]
Now we write the equation for $\eta$ as follows:
\[
\dot{\eta} = \mu \eta + T_0 \eta \alpha_0 + T_1 \eta \alpha_2 + T_2 \eta \beta_2 + T_3 \eta^3 + O(\delta^4),
\]
where $\mu$ is the critical eigenvalue which satisfies $\mu = 0$ at the critical point and
the constants are:
\[
T_0 = \frac{f_u - Df_v}{f_u - D^2f_v} \{f_{uu} - Df_{uv} - (f_{uv} - Df_{vv})/\tau \},
\]
\[
T_1 = \frac{f_u - Df_v}{f_u - D^2f_v} (f_{uu} - Df_{uv}),
\]
\[
T_2 = \frac{f_u - Df_v}{f_u - D^2f_v} (f_{uv} - Df_{vv}),
\]
\[
T_3 = \frac{f_u - Df_v}{2(f_u - D^2f_v)} (f_{uuu} - 3f_{uvu}D + 3f_{uvv}D^2 - f_{vvv}D^3).
\]

Next, we obtain the equation for the cooperate modes similarly as follows.
\[
\dot{\alpha}_0 = (f_u - \frac{f_v}{\tau})\alpha_0 + 2N_2\eta^2 + O(\delta^3),
\]
\[
\left( \begin{array}{c}
\dot{\alpha}_2 \\
\dot{\beta}_2
\end{array} \right) = A_2 \left( \begin{array}{c}
\alpha_2 \\
\beta_2
\end{array} \right) + N_2\eta^2 \left( \begin{array}{c}
1 \\
-1
\end{array} \right) + O(\delta^3),
\]
where \( N_2 = (f_{uu} - 2Df_{uv} + D^2f_{vv})/2. \)

Now we find a suitable near-identity transformation for the critical mode \( \eta \)
\[
\xi = \eta - S_0\eta\alpha_0 - S_1\eta\alpha_2 - S_2\eta\beta_2,
\]
so that the equation for the critical mode can be written only by the critical mode. It is the so-called normal form transformation by which we obtain the reduced equation on the center manifold. Our task here is to determine the unknown constants \( S_0, S_1, S_2 \) in order that the terms including slave modes vanish. By using the inverse transformation of the local diffeomorphism (4.13)
\[
\eta = \xi + S_0\xi\alpha_0 + S_1\xi\alpha_2 + S_2\xi\beta_2 + O(\xi^4),
\]
we obtain the transformed equation for \( \xi \):
\[
\dot{\xi} = \mu\xi
\]
\[
-2\mu S_0\xi\alpha_0 - 2\mu S_1\xi\alpha_2 - 2\mu S_2\xi\beta_2
\]
\[
+ T_0\xi\alpha_0 + T_1\xi\alpha_2 + T_2\xi\beta_2
\]
\[
- S_0(f_u - \frac{f_v}{\tau})\xi\alpha_0 - \xi(S_1 \quad S_2)A_2 \left( \begin{array}{c}
\alpha_2 \\
\beta_2
\end{array} \right)
\]
\[
- N_2(2S_0 + S_1 - \frac{S_2}{\tau})\xi^3 + T_3\xi^3 + O(\delta^4).
\]
Since it is sufficient to consider the critical case \( \mu = 0 \), the unknown constants are obtained by solving the following equations:
\[
(f_u - \frac{f_v}{\tau})S_0 = T_0,
\]
\[
(S_1 \quad S_2)A_2 = (T_1 \quad T_2)
\]
A simple calculation shows that \( \det A_2 = \frac{12}{\tau D} (f_u - Df_v)^2 \) and
\[
2S_0 + S_1 - \frac{S_2}{\tau} = 2 \left( f_u - \frac{f_v}{\tau} \right)^{-1} T_0 + \frac{DT_2 - T_1}{3(f_u - Df_v)}.
\]

In consequence we have the cubic coefficient \( c_1 \) of (4.14) explicitly as
\[
c_1 = T_3 - N_2 \left\{ 2 \left( f_u - \frac{f_v}{\tau} \right)^{-1} T_0 + \frac{DT_2 - T_1}{3(f_u - Df_v)} \right\}.
\] (4.15)

The next theorem tells the local bifurcation analytically.

**Theorem 4.1** Suppose \( c_1 \) is not zero at the constant equilibrium \((\overline{u}, \overline{v})\) then the dynamics on the center manifold of the equation (4.5) near the critical point \( k_0 = k^* \) is given by
\[
\dot{\xi} = \mu \xi + c_1 \xi^3 + O(\delta^4).
\]

By virtue of this theorem we can assert that the nontrivial stationary solution emerges in supercritical pitchfork bifurcation when \( c_1 \) is negative.

**Remark 4.1** Not only \( k_0 \) but also other constant such as \( s \) can be considered as a bifurcation parameter if \( \frac{\partial \mu}{\partial \sigma} \neq 0 \). Moreover, we can see that the same type of bifurcation occurs as such a parameter varies.

**Remark 4.2** As \( D \) tends to zero, the limit of \( c_1 \) gives a simpler criterion for the pitchfork bifurcation:
\[
\lim_{D \to 0} c_1 = \frac{f_{uu}}{2} - \frac{(\tau f_{uu} - f_{uv}) f_{uu}}{\tau f_u - f_v} + \frac{f_{uu}^2}{6f_u} \quad (4.16)
\]

Let us determine the cubic coefficient \( c_1 \) of the normal form in the case of (1.8):
\[
f(u, v) = a_1(u + v)[(\alpha u + v)(u + v) - a_2].
\]

Suppose \( \tau = 1 \) then \( \overline{u} + \overline{v} = s \). Since \( f(\overline{u}, \overline{v}) = 0 \) means \((\alpha \overline{u} + \overline{v})(\overline{u} + \overline{v}) = a_2 \), i.e., \((\alpha \overline{u} + \overline{v}) = a_2/s\), we can calculate the partial derivatives of \( f \) at \((\overline{u}, \overline{v})\) explicitly. Then we have
\[
\lim_{D \to 0} c_1 = 3a_1 \alpha - \frac{2a_1(4\alpha s^2 + 5a_2)(2\alpha s^2 + a_2)}{3s^2(\alpha s^2 + a_2)}.
\]

We calculate this value with the particular set of parameters given in the previous sub-section so that
\[
\lim_{D \to 0} c_1 = 3 \cdot 0.5 \cdot 0.1 - \frac{2 \cdot 0.5(4 \cdot 0.1 \cdot 10^2 + 5 \cdot 2.2)(2 \cdot 0.1 \cdot 10^2 + 2.2)}{3 \cdot 10^2(0.1 \cdot 10^2 + 2.2)} = -0.1836 \cdots,
\]
therefore, we see that the pitchfork bifurcation observed in the case of (1.8) is supercritical for sufficiently small $D$.

We can similarly evaluate $c_1$ in the case of (1.7) as follows.

$$\lim_{D \to 0} c_1 = \frac{a_1 a_2^2 (11a_2^2 s^2 - 44a_2 s + 35)}{3(a_2 s - 1)(a_2 s + 1)^5}$$

It turns out that for $s > 1/a_2 \approx 1.429$ there are two critical values: $s = s_1 \approx 1.565$ and $s = s_2 \approx 4.1493$ where the sign of $c_1$ changes. It is consistent to the results shown in Figure 2.

## 5 Appendix

Here we give a condition on $f$ under which the semiflow $S(t)$ defined by (1.10) is $C^k$ in $X = H^1(\Omega) \times H^1(\Omega)$.

First let $\mathcal{A}$ in $L^2(\Omega) \times L^2(\Omega)$ be a closed operator defined by

$$\mathcal{A}u := -\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \Delta u + u,$$

with the domain

$$\mathcal{D}(\mathcal{A}) := \{u \in H^2(\Omega) \times H^2(\Omega) : \partial u / \partial \nu = 0 \text{ on } \partial \Omega\}$$

or

$$\mathcal{D}(\mathcal{A}) := \{u \in H^2(0, L) \times H^2(0, L) : u(0) = u(L), u_x(0) = u_x(L)\}.$$ 

Then $\mathcal{A}$ is sectorial and it allows a fractional power $\mathcal{A}^\alpha$ ($0 \leq \alpha < 1$). It is known that

$$\mathcal{D}(\mathcal{A}^{1/2}) = H^1(\Omega) \times H^1(\Omega),$$

or

$$\mathcal{D}(\mathcal{A}^{1/2}) = \{u \in H^1(0, L) \times H^1(0, L) : u(0) = u(L)\}.$$ 

By the existence theorem found in [9] we see that a solution map

$$(u_0(\cdot), v_0(\cdot)) \mapsto (u(\cdot, t; u_0, v_0), v(\cdot, t; u_0, v_0))$$

is $C^k$ in $H^1(\Omega) \times H^1(\Omega)$ if

$$f(\cdot, v) \in H^1(\Omega) \times H^1(\Omega) \mapsto f(u(\cdot, v)) \in L^2(\Omega) \times L^2(\Omega) \quad (5.1)$$

is $C^k$. For the one-dimensional case, since $C^0[0, L]$ is imbedded in $H^1([0, L])$, $f$ of (5.1) is $C^k$ provided that $f(u, v)$ is a $C^k$ function. As for the higher-dimensional case, we have the following lemma (given in Hale-Vegas[8]):
Lemma 5.1 Let $\Omega$ is a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with smooth boundaries. Suppose that $f(u,v)$ is $C^1$ and it has zero. If the partial derivatives $f_u$ and $f_v$ are uniformly bounded in $\mathbb{R}^2$, then $f$ of (5.1) is $C^1$.

Proof. For simplicity of notation we consider the case that $f = f(u)$. Suppose

$$\sup_{u \in \mathbb{R}} |f'(u)| < \infty.$$ 

Then the map $\hat{f}$

$$\hat{f} : H^1(\Omega) \longrightarrow L^2(\Omega)$$

makes sense since $f(u)$ has zero. We prove that the Gataux derivative $D_G \hat{f}(u) (u = u(\cdot) \in H^1(\Omega))$

$$D_G \hat{f}(u) : w(\cdot) \in H^1(\Omega) \mapsto f'(u(\cdot))w(\cdot) \in L^2(\Omega)$$

is continuous with respect to $u \in H^1(\Omega)$. Then it turns that $\hat{f}$ is Fréchet continuously differentiable and the derivative coincides with $D_G \hat{f}$. To prove the continuity of $D_G \hat{f}(u)$, let $\{u_j\}$ be any sequence in $H^1(\Omega)$ which converges to $u$ and estimate

$$\|D_G \hat{f}(u_j) - D_G \hat{f}(u)\|_{L(H^1(\Omega); L^2(\Omega))}. \quad (5.2)$$

There exists a subsequence $\{u_{j_i}\}$ such that $u_{j_i}$ converges to $u$ a.e. By the Schwarz inequality and the imbedding $H^1(\Omega) \subset L^{2n/(n-2)}(\Omega)$ (if $n = 2$, $H^1(\Omega) \subset L^q(\Omega)$ for $q \geq 2$), we obtain

$$\|\{D_G \hat{f}(u_{j_i}) - D_G \hat{f}(u)\}(w)\|_{L^2} \leq \|f'(u_{j_i}(\cdot)) - f'(u(\cdot))\|_{L^p}\|w\|_{L^q} \leq C\|f'(u_{j_i}(\cdot)) - f'(u(\cdot))\|_{L^p}\|w\|_{H^1},$$

where we take $p = n, q = 2n/(n - 2)$ for $n \geq 3$. Since $|f'|$ is bounded, the Lebesgue convergence theorem tells

$$\lim_{j_i \to \infty} \int_{\Omega} |f'(u_{j_i}(x)) - f'(u(x))|^p dx = \int_{\Omega} \lim_{j_i \to \infty} |f'(u_{j_i}(x)) - f'(u(x))|^p dx = 0.$$

Since $\{u_j\}$ is an arbitrary sequence, we obtain the continuity of $D_G \hat{f}(u)$. This concludes the proof. \qed

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